Proposal of modified Kalman filter incorporating stochastic dynamic analysis in prediction procedure

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Abstract. This paper proposes the modified Kalman filter in order systematically to deal with a nonlinear system and a parametric uncertainty excitation system. Conventional Kalman type filtering problem for linear system with parametric uncertainties is based on a linear-Gaussian process. Therefore, this type state estimator cannot be applied to the nonlinear system. Hence, the Kalman filter methodology that can systematically account for the nonlinear and the parametric uncertainty excitation systems is desirable. The present study proposes a modified Kalman filter incorporating stochastic dynamic analysis (KF-SDA) in the prediction step to deal with the abovementioned systems systematically. The proposed KF-SDA algorithm will be derived using the solution of moment equation. Moreover, a fundamental verification will be performed for a single degree-of-freedom (DOF) system and two-DOF system subjected to white noise excitation. The state estimation of the single-DOF system, which is simultaneously subjected to white noise and stochastic parametric excitations, will be considered. Furthermore, the state estimation of the nonlinear single-DOF system subjected to the white noise excitation will be performed.

1. Introduction

The filtering problem is significant challenge in various industrial areas, such as aerospace, signal processing of precise instruments, and agriculture. Classical filtering was introduced by N. Wiener in the 1940s [1]. Twenty years later, Kalman and Bucy achieved successful implementation of filtering in digital processing [2,3]. In particular, considering the modeling uncertainty is important for the precise evaluation of an un-observed state vector. In fact, many studies have investigated the Kalman filter considering the modeling uncertainty [4,5,6]. The modeled system parameters by random perturbations based on stochastic models with a time-dependent noise (or multiplicative noise).

A Kalman-type filtering problem for a linear system with parametric uncertainties is presented [6,7,8]. In this work, a Luenberger observer-type filter was derived in a continuous-time stochastic system [7]. Furthermore, the Kalman-type filtering problem for a linear system subject to parametric uncertainty excitation was expanded to a discrete-time system [8]. These previous works conducted benchmark tests with the classical Kalman filter. Subsequently, the accuracy in these works was higher than that obtained using its classical counterpart.

However, this type of state estimator cannot be applied to a nonlinear system, because it is based on a linear Gaussian process. Therefore, the Kalman filter methodology that can systematically account for the nonlinear and parametric uncertainty excitation systems is desirable.

The present study proposes a modified Kalman filter incorporating stochastic dynamic analysis (KF-SDA) in the prediction step to deal with the abovementioned systems systematically. The moment equation of probabilistic dynamics is described as the stochastic dynamics consisting of nonlinear / non-Gaussian characteristics. The methodology between the nonlinear and parametric uncertainty

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excitation systems will be realized systematically using the solution of moment equation in the prediction procedure of a conventional linear-Gaussian Kalman filter.

The proposed KF-SDA algorithm will be derived using the solution of moment equation. Moreover, a fundamental verification will be performed for a single degree-of-freedom (DOF) system and two-DOF system subjected to white noise excitation. The state estimation of the single-DOF system, which is simultaneously subjected to white noise and stochastic parametric excitations, will be considered. Furthermore, the state estimation of the nonlinear single-DOF system subjected to the white noise excitation will be performed.

2. Modified Kalman filter using solution of moment equation

The KF-SDA is obtained by replacing the formula of the prediction process in the conventional Kalman filter. The prediction step in the conventional Kalman filter is composed of the update state estimation value and update error covariance matrices. The update formula is based on the analytical solution of the Ito probabilistic differential equation of a linear system. Contrary to the previous Kalman filter, replacing the Ito probabilistic differential equation with the moment equation is considered in the prediction procedure herein.

The Ito probabilistic differential equation is represented by Eq. (1).

$$\frac{d\boldsymbol{x}}{dt} = \phi(\boldsymbol{x}, t) + \gamma(\boldsymbol{x}, t)\boldsymbol{W}(t), \tag{1}$$

where, $\boldsymbol{x}$ represents the state vector of $1 \times n$ size and $\phi(\boldsymbol{x}, t)$ represents the system dynamics. For example, the system dynamics in the linear case is $\phi(x, t) = A\boldsymbol{x}(t)$. The right hand second term of $\gamma(\boldsymbol{x}, t)\boldsymbol{W}(t)$ represents the white noise excitation. The moment equation is as follows:

$$\frac{dE[h]}{dt} = \sum_{i=1}^{n} E\left[\phi_i\left(\frac{\partial h}{\partial x_i}\right)\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)[\gamma D\gamma^T]_{i,j}\right] \tag{2}$$

The moment equation represents the time evolution of each moment. Here, the right-hand first term represents the contribution from the first-order moment, while the right-hand second term represents the contribution from the second-order moment. $E[\cdot]$ represents the expected values for calculation. h denotes the products of the probabilistic variables. For example, $h = x_1^i x_2^j \dots x_n^k$. Furthermore, the notation for the moment of $m_{ij\dots k}$ is introduced to obtain a simple description (i.e. $m_{i,j,\dots k} = E[h]$). The index of $i, j, \dots, k$ has the constraints by the moment order. For example, in case of the first-order moment, the constraint is $i + j + \dots + k = 1$(i.e., the index combinations are $i = 1, j = 0, \dots, k = 0$, $i = 0, j = 1, \dots, k = 0$, and $i = 0, j = 0, \dots, k = 1$). Moreover, D represents the diffusion coefficient. The moment equation is numerically solved by the Runge-Kutta method.

Fig.1 shows the algorithmic flowchart of the KF-SDA. At first, the proposed KF-SDA required initial values in terms of moment, diffusion coefficient of the system noise, observation model, and variance of the observation noise. The state vector was defined based on the initial value of the moment and update state vector. The state vector was identified as the first-order moment. In the prediction step, the moment equation was numerically solved by the Runge-Kutta method. The prior state vector of $\boldsymbol{x}^-$ and prior covariance of P^- were obtained by solving the moment equation. Furthermore, in the filtering step, the Kalman gain was calculated based on the procedure of the ordinal linear Kalman filter. The prior state estimation values were updated by the residual between the observation time series of $y(t)$, prediction values of $\boldsymbol{cx}^-$ and observation model of $\boldsymbol{c}$. Moreover, the prior covariance of P^- was updated by the Kalman gain and observation model.

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The procedure of proposed KF-SDA was realized by alternating between the prediction step and filtering steps.

START

Input the initial value of moment $m_{ij...k}(0)$, diffusion coefficients of system noise D_{ij}, Observation model $\boldsymbol{c}$ and Observation noise variance R

Prediction step
Solve the "Moment equation" using by Runge-Kutta method

$$\frac{dE[h]}{dt} = \sum_{i=1}^{n} E\left[\phi_i\left(\frac{\partial h}{\partial x_i}\right)\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)[\gamma D \gamma^T]_{i,j}\right]$$

Prior state estimation value : $\boldsymbol{x}^- = [\dots m_{ij\dots k} \dots]^T, (i+j+\dots+k=1)$

Prior covariance : $P^- = \begin{bmatrix} m_{ij\dots k} \end{bmatrix}, (i+j+\dots+k=2)$

Filtering step
Kalman gain : $K = P^- \boldsymbol{c}^T (\boldsymbol{c} P^- \boldsymbol{c}^T + R)^{-1}$
Update of state vector : $\boldsymbol{x} = \boldsymbol{x}^- + K(y(t) - \boldsymbol{c}\boldsymbol{x}^-)$
Update of covariance matrix : $P = (I - K\boldsymbol{c})P^-$
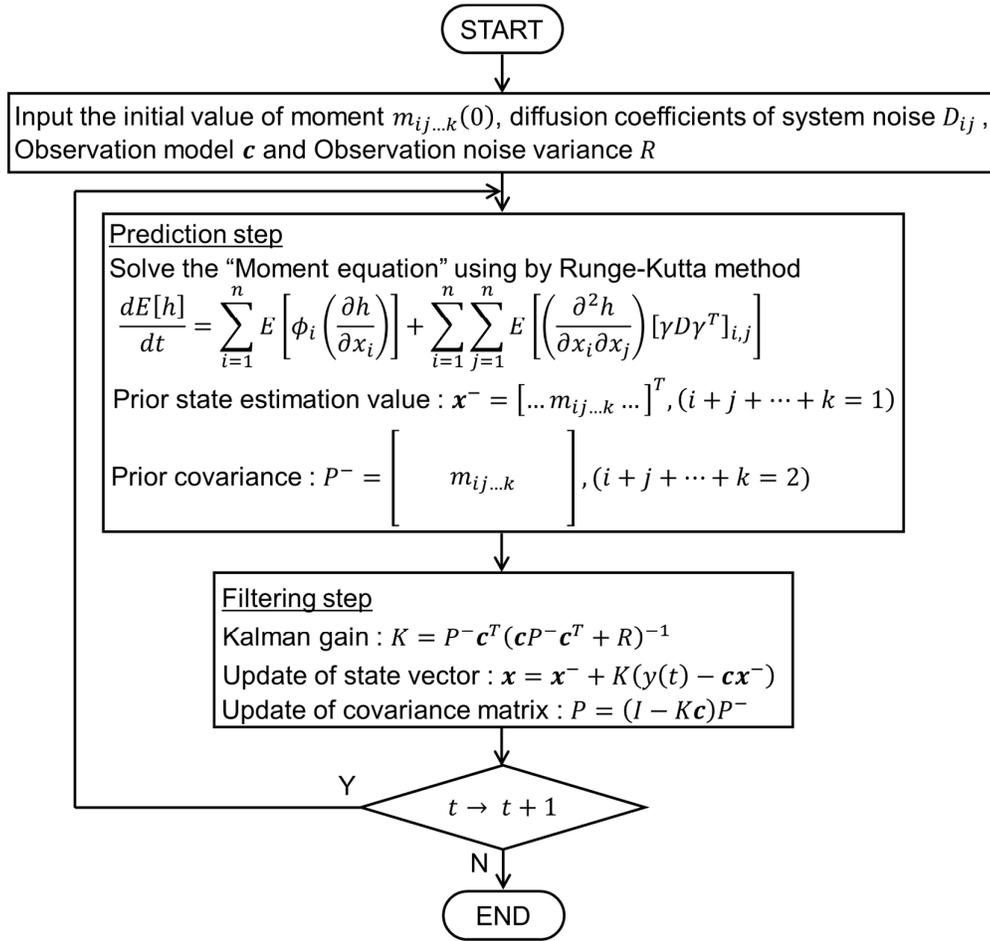
Y $\quad t \to t+1$

N

END

Fig. 1 Algorithmic flowchart of the proposed modified Kalman filter (KF-SDA).

3. Numerical simulation for the fundamental operation verification

In this section, we will conduct numerical simulations using the proposed KF-SDA. The numerical simulations consider two examples. First, the fundamental operation verification is conducted using a linear single-DOF system subjected to white noise excitation. Second, the extensibility to the multi-DOF system is verified using a linear two-DOF system subjected to white noise excitation.

3.1 Single-DOF system subjected to white noise excitation

3.1.1 Details of the formulation

The governing equation of the single-DOF system in the case of a unit mass, subjected to white noise excitation, is as follows:

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = w, \tag{3}$$

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where, x represents the displacement; c represents the damping coefficient; k represents the spring constant; and w represents the white noise excitation force. Eq. (3) is rewritten to the form of the Ito probabilistic differential equation as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}w, \quad \phi = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{4}$$

where, x_1 and x_2 represent the displacement and velocity, respectively. Therefore, the moment equation is derived as follows:

Moment equation in terms of the first-order moment

$$\frac{d}{dt}\begin{bmatrix} m_{10} \\ m_{01} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}\begin{bmatrix} m_{10} \\ m_{01} \end{bmatrix} \tag{5}$$

Moment equation in terms of the second-order moment

$$\frac{d}{dt}\begin{bmatrix} m_{20} \\ m_{11} \\ m_{02} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ -k & -c & 1 \\ 0 & -2k & -2c \end{bmatrix}\begin{bmatrix} m_{20} \\ m_{11} \\ m_{02} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2D \end{bmatrix} \tag{6}$$

Here, m_{10} represents the first-order moment of $m_{10} = E[x_1^1 x_2^0] = E[x_1]$, m_{01} represents the first-order moment of $m_{01} = E[x_1^0 x_2^1] = E[x_2]$, m_{20} represents the second-order moment of $m_{20} = E[x_1^2 x_2^0] = E[x_1^2]$, m_{11} represents the second-order moment of $m_{20} = E[x_1^1 x_2^1] = E[x_1 x_2]$ and m_{02} represents the second-order moment of $m_{02} = E[x_1^0 x_2^2] = E[x_2^2]$. Eqs. (5) and (6) are numerically solved by the Runge-Kutta method. Here, the observation time series is supposed to displacement signal. The observation model of $\boldsymbol{c}$ is

$$\boldsymbol{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{7}$$

The prior state vector of $\boldsymbol{x}^-$ and prior covariance of P^- are represented in Eq. (8) as.

$$\boldsymbol{x}^- = \begin{bmatrix} m_{10} \\ m_{01} \end{bmatrix}, \quad P^- = \begin{bmatrix} m_{20} & m_{11} \\ m_{11} & m_{02} \end{bmatrix} \tag{8}$$

3.1.2 Result of the state estimation

In this section, the estimation experiment for the non-observed velocity is conducted under the displacement measurement. The true displacement and velocity values are obtained by solving the equation of motion (Eq. (3)) using the Runge-Kutta method.

The calculation conditions are as follows: spring constant $k = 1$; damping constant $c = 0.05$; variance of white system noise $\sigma_v^2 = 1$; and time increment for the Runge-Kutta method $\Delta t = 0.1$. Additionally, the initial conditions are $x(0) = 0$ and $v(0) = 0$. The true values of the displacement and the velocity of each oscillator are represented by the solid black line in Fig. 2. Fig. 2(a) shows the state estimation result for the displacement, while Fig. 2(b) depicts that for the velocity. The state estimation conditions are $R = 10^{-1}$ and $D = 0.05$. Furthermore, the initial conditions of each moment are as follows: $m_{10}(0) = 0$; $m_{01}(0) = 0$; $m_{20}(0) = 1$; $m_{11}(0) = 0$; and $m_{02}(0) = 1$.

The solid blue line in Fig. 2 depicts the state estimation results for the displacement and velocity. Additionally, the estimation values of both displacement and velocity were in good agreement with the true values. Here, correlation coefficient between the true signal X and the estimated signal Y

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defines as the $\rho(X,Y) = E[(X - E[X])(Y - E[Y])]/\sqrt{E[(X - E[X])^2]E[(Y - E[Y])^2]}$. Then, the correlation coefficients between the true and estimated signals were 0.9812 and 0.9613 for the displacement and velocity, respectively. The correspondence between the true values and the estimation values is confirmed in the case of differ initial condition of KF-SDA (for example, the case of $m_{10}(0) = 0.3$, $m_{01}(0) = 0$, the case of $m_{10}(0) = 0$, $m_{01}(0) = 0.3$ and the case of $m_{10}(0) = 0.3$; $m_{01}(0) = 0.3$, and so on).

(a) Displacement   (b) Velocity
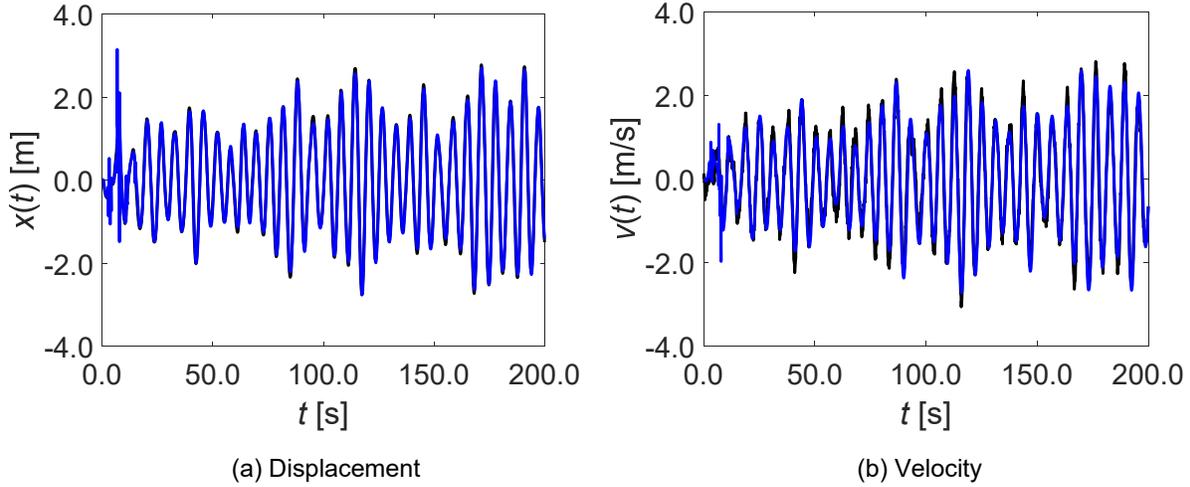
Fig. 2 State estimation results for displacement and velocity using the proposed KF-SDA in the single-DOF system subjected to white noise excitation.

3.2 Two-DOF system subjected to white noise excitation

3.2.1 Details of the formulation

The governing equation of the two-DOF system in the case of the unit mass subjected to white noise excitation is as follows:

$$\begin{aligned}
\frac{d^2u_1}{dt^2} + ku_1 + k(u_1 - u_2) - 2c\frac{du_1}{dt} + c\frac{du_2}{dt} &= w, \\
\frac{d^2u_2}{dt^2} + ku_2 + k(u_2 - u_1) - 2c\frac{du_2}{dt} + c\frac{du_1}{dt} &= 0.
\end{aligned} \tag{9}$$

Where, u_1 and u_2 represent the displacements of first and second oscillators, respectively; c represents the damping coefficient; k represents the spring constant; and w represents the white noise excitation force.

Eq. (9) is rewritten to the form of the Ito probabilistic differential equation, as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -2kx_1 + kx_3 - 2cx_2 + cx_4 \\ x_4 \\ kx_1 - 2kx_3 + cx_2 - 2cx_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} w, \tag{10}$$

where, x_1 and x_2 represent the displacement and velocity of the first oscillator, respectively, and x_3 and x_4 represent the displacement and velocity of the second oscillator, respectively. Therefore, the moment equation is derived as follows:

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Moment equation in terms of the first-order moment

$$\frac{d}{dt}\begin{bmatrix} m_{1000} \\ m_{0100} \\ m_{0010} \\ m_{0001} \end{bmatrix} = \begin{bmatrix} m_{1000} \\ -2km_{1000} - 2cm_{0100} + km_{0010} + cm_{0001} \\ m_{0001} \\ km_{1000} + cm_{0100} - 2km_{0010} - 2cm_{0001} \end{bmatrix} \tag{11}$$

Moment equation in terms of the second-order moment

$$\frac{d}{dt}\begin{bmatrix} m_{2000} \\ m_{1100} \\ m_{1010} \\ m_{1001} \\ m_{0200} \\ m_{0110} \\ m_{0101} \\ m_{0020} \\ m_{0011} \\ m_{0002} \end{bmatrix} = \begin{bmatrix} 2m_{1100} \\ m_{0200} - 2km_{2000} + km_{1010} + cm_{1001} - 2cm_{1100} \\ m_{0110} + m_{1001} \\ m_{0101} + km_{2000} - 2km_{1010} + cm_{1100} - 2cm_{1001} \\ -4km_{1100} + 2km_{0110} + 2cm_{0101} - 4cm_{0002} \\ -2km_{1010} + km_{0020} + cm_{0011} - 2cm_{0110} + m_{0101} \\ -2km_{1001} + km_{0011} + cm_{0002} - 4cm_{0101} + km_{1100} - 2km_{0110} + cm_{0200} \\ 2m_{0011} \\ m_{0002} + km_{1010} - 2km_{0020} + cm_{0110} - 2cm_{0011} \\ 2km_{1001} - 4km_{0011} + 2cm_{0101} - 4cm_{00002} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2D \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

Eqs. (11) and (12) are numerically solved by the Runge-Kutta method. Here, the observation time series is displacement signal of the first oscillator. The observation model of c is as follows:

$$c = [1 \quad 0 \quad 0 \quad 0]. \tag{13}$$

The prior state vector of x^- and the prior covariance of P^- are represented in Eq. (14) as.

$$x^- = \begin{bmatrix} m_{1000} \\ m_{0100} \\ m_{0010} \\ m_{0001} \end{bmatrix}, \quad P^- = \begin{bmatrix} m_{2000} & m_{1100} & m_{1010} & m_{1001} \\ m_{1100} & m_{0200} & m_{0110} & m_{0101} \\ m_{1010} & m_{0110} & m_{0020} & m_{0011} \\ m_{1001} & m_{0101} & m_{0011} & m_{0002} \end{bmatrix} \tag{14}$$

3.2.2 Results of the state estimation

In this section, the estimation experiments for the non-observed displacement of the second oscillator and non-observed velocity of the two oscillators are conducted under the displacement measurement of the first oscillator. The true values of the displacement and velocity signals are obtained by solving the equation of motion (Eq. (9)) using the Runge-Kutta method.

The calculation conditions are as follows: spring constant $k = 1$; the damping constant $c = 0.01$; variance of white system noise $\sigma_v^2 = 1$; and time increment for the Runge-Kutta method $\Delta t = 0.1$. The initial conditions are $x_1(0) = x_2(0) = 0$ and $v_1(0) = v_2(0) = 0$. The true values of the displacement and the velocity of each oscillator are represented by the solid black line in Fig. 3. Fig. 3(a) shows the state estimation result for the displacement of the first oscillator, while (b) depicts that for the displacement of the second oscillator. Fig.3 (c) illustrates the state estimation result for the velocity of the first oscillator, while Fig. 3(d) displays that for the second oscillator. The state estimation conditions are $R = 10^{-1}$ and $D = 0.05$. The initial conditions of each moment are: $m_{1000}(0) = 0$; $m_{0100}(0) = 0$; $m_{0100}(0) = 0$; $m_{0001}(0) = 0$; $m_{2000}(0) = 1$; $m_{1100}(0) = 0$; $m_{1010}(0) = 0$; $m_{1001}(0) = 0$; $m_{0200}(0) = 1$; $m_{0110}(0) = 0$; $m_{0101}(0) = 0$; $m_{0020}(0) = 1$; $m_{0011}(0) = 0$; and $m_{0002}(0) = 1$.

The estimation results for the displacement and velocity are represented by the solid blue line in Fig. 3. Additionally, the estimation values of both displacement and velocity were in good agreement

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with the true values. The correlation coefficients between the true and estimated signals were 0.9843 for the displacement of the first oscillator, 0.9800 for the velocity of the first oscillator, 0.9247 for the displacement of the second oscillator, and 0.9741 for the velocity of the second oscillator.
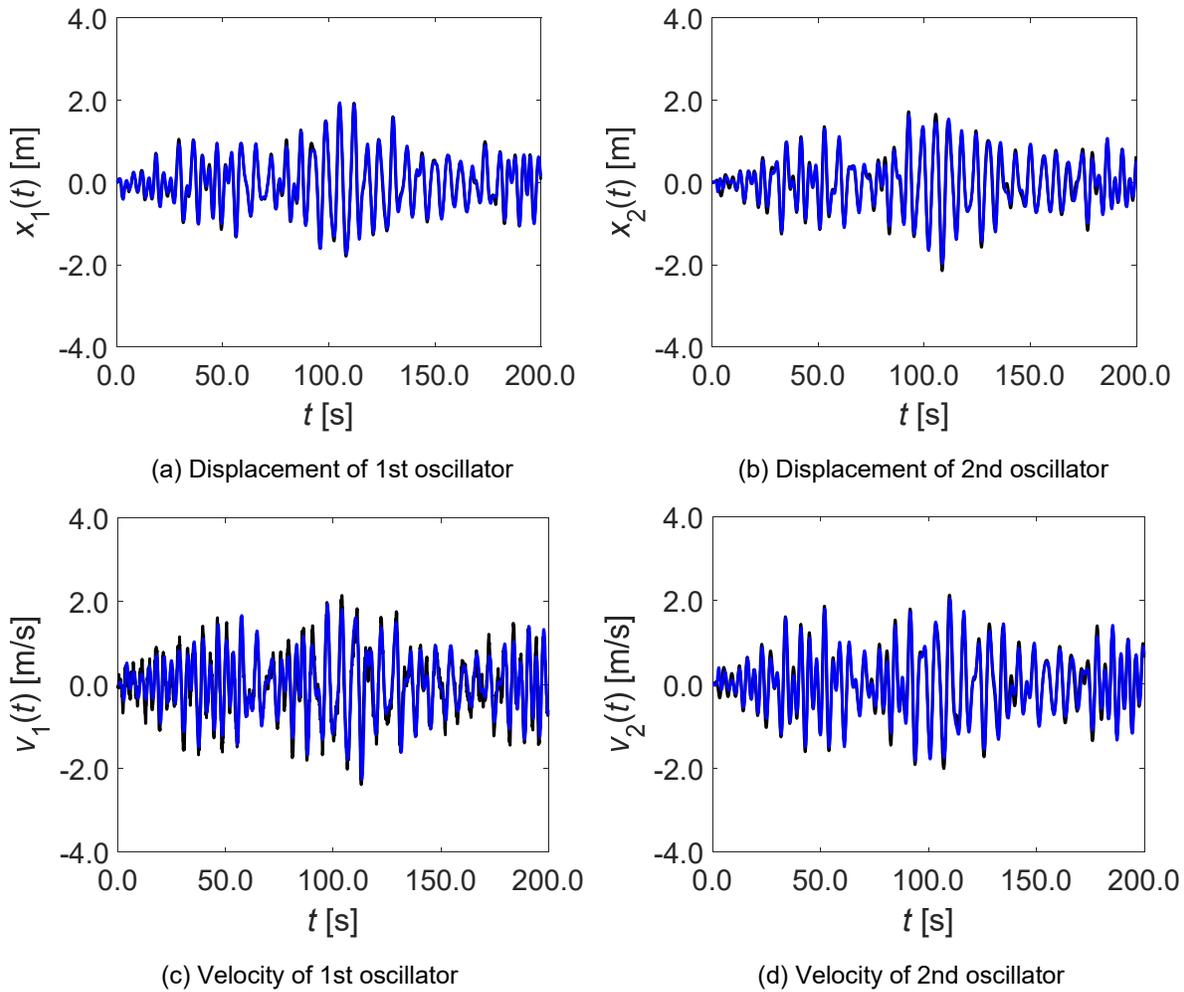

(a) Displacement of 1st oscillator

(b) Displacement of 2nd oscillator

(c) Velocity of 1st oscillator

(d) Velocity of 2nd oscillator

Fig. 3 State estimation results for displacement and velocity using the proposed KF-SDA in case of the two-DOF system subjected to white noise excitation.

4. Application to the single-DOF system simultaneously subjected to white noise and random parametric excitations

4.1 Details of the formulation

The governing equation of the single-DOF system in the case of the unit mass simultaneously subjected to the random parametric and white noise excitations is

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + (k + w_1)x = w_2, \tag{15}$$

where, x represents the displacement; c represents the damping coefficient; k represents the spring constant; w_1 represents the white random parametric excitation force; and w_2 represents the white noise excitation force.

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Eq. (15) is rewritten to the form of the Ito probabilistic differential equation as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -cx_2 - kx_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -x_1 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{16}$$

where, x_1 and x_2 represent the displacement and the velocity, respectively. Therefore, the moment equation is derived as follows:

Moment equation in terms of the first-order moment

$$\frac{d}{dt}\begin{bmatrix} m_{10} \\ m_{01} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}\begin{bmatrix} m_{10} \\ m_{01} \end{bmatrix}. \tag{17}$$

Moment equation in terms of the second-order moment

$$\frac{d}{dt}\begin{bmatrix} m_{20} \\ m_{11} \\ m_{02} \end{bmatrix} = \begin{bmatrix} 2m_{11} \\ -km_{20} - cm_{11} + m_{02} \\ -2km_{11} - 2cm_{02} + 2D_{11}m_{20} - 4D_{12}m_{10} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2D_{22} \end{bmatrix}. \tag{18}$$

Eqs. (17) and (18) are numerically solved by the Runge-Kutta method. Here, the observation time series is displacement signal. The observation model of c is as follows:

$$c = [1 \quad 0]. \tag{19}$$

The prior state vector of x^- and prior covariance of P^- are represented in Eq. (20) as

$$x^- = \begin{bmatrix} m_{10} \\ m_{01} \end{bmatrix}, \quad P^- = \begin{bmatrix} m_{20} & m_{11} \\ m_{11} & m_{02} \end{bmatrix}. \tag{20}$$

4.2 Results of the state estimation

In this section, the estimation experiment for the non-observed velocity is conducted under the displacement measurement. The true values of the displacement and the velocity are obtained by solving the equation of motion (Eq. (15)) using the Runge-Kutta method. The calculation conditions are: spring constant $k = 1$; damping constant $c = 0.05$; variance of white parametric random excitation $\sigma_1^2 = 10^{-1}$; variance of white system noise $\sigma_2^2 = 1$; and time increment for the Runge-Kutta method $\Delta t = 0.1$. The initial conditions are $x(0) = 0$ and $v(0) = 0$.

The true values of the displacement and the velocity are represented by the solid black line in Fig. 4. Fig. 4(a) shows the state estimation result for the displacement, while Fig.4(b) depicts that for the velocity. The state estimation conditions are: $R = 10^{-1}$; $D_{11} = 5 \times 10^{-4}$; $D_{12} = 0.05$. Furthermore, the initial conditions of each moment are: $m_{10}(0) = 0$; $m_{01}(0) = 0$; $m_{20}(0) = 1$; $m_{11}(0) = 0$; and $m_{02}(0) = 1$.

The estimation results for the displacement and the velocity are represented by the solid blue line in Fig. 4. Additionally, the estimation values of both displacement and velocity are in good agreement with the true values. The correlation coefficients between the true and estimated signals were 0.9904 and 0.9819 for the displacement and the velocity, respectively.

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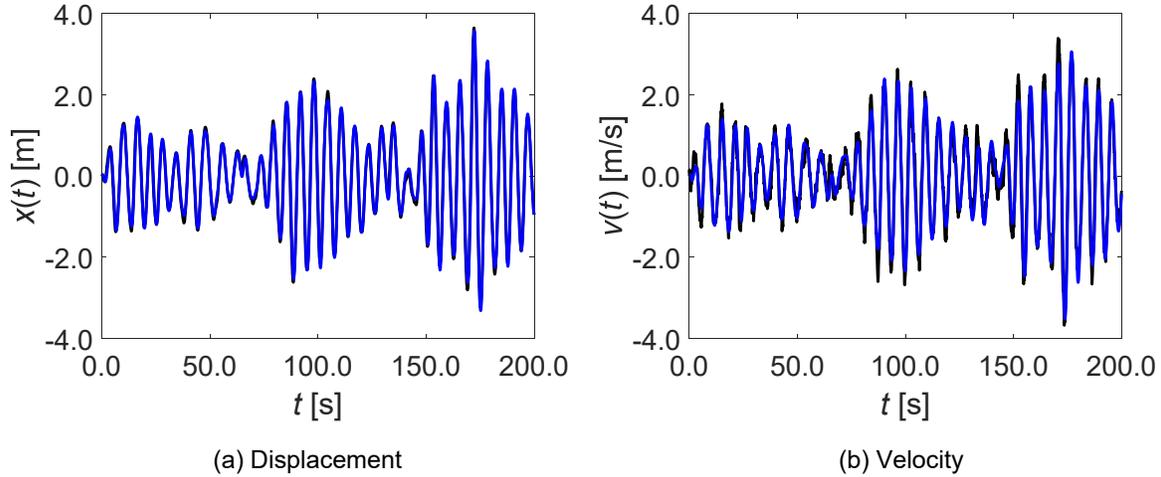
(a) Displacement   (b) Velocity

Fig. 4 State estimation results for displacement and velocity using the proposed KF-SDA in the single-DOF system simultaneously subjected to white noise and random parametric excitations.

5. Application to the Duffing system subjected to white noise excitation

5.1 Details of the formulation

The government equation of the Duffing system in the case of the unit mass subjected to white noise excitation is as follows:

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx + \mu x^3 = w, \tag{21}$$

where, x represents the displacement; c represents the damping coefficient; k represents the spring constant; μ represents the nonlinear spring coefficient; and w represents the white noise excitation force. Eq. (21) is rewritten to the form of the Ito probabilistic differential equation as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -kx_1 - \mu x_1^3 - cx_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \tag{22}$$

where, x_1 and x_2 represent displacement and velocity, respectively. Therefore, the moment equation is derived as follows:

Moment equation in terms of the first-order moment

$$\frac{d}{dt}\begin{bmatrix} m_{10} \\ m_{01} \end{bmatrix} = \begin{bmatrix} m_{01} \\ -km_{10} - cm_{01} \end{bmatrix}. \tag{23}$$

Moment equation in terms of the second-order moment

$$\frac{d}{dt}\begin{bmatrix} m_{20} \\ m_{11} \\ m_{02} \end{bmatrix} = \begin{bmatrix} 2m_{11} \\ m_{02} - km_{20} - 3\mu m_{20}^2 - cm_{11} \\ -2km_{11} - 2cm_{02} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2D \end{bmatrix}. \tag{24}$$

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Eq. (24) contains the third-order moment term of m_{30} in spite the second-order moment equation. Thus, the moment equation cannot solve the numerically and analytically, the characteristics so called the moment closure. The approximation is required the retaining the moment closure. Here, m_{30} was neglected in the derivation process of the first-order moment. Moreover, the odd-th higher moments were neglected as well, and the even-th higher moments approximated the relationship of $m_{2n} = 1 \cdot 3 \cdots (2n - 1)m_2^n$.

Eqs. (23) and (24) are numerically solved by the Runge-Kutta method. The observation time series is displacement signal. The observation model of c is presented as follows:

$$c = [1 \quad 0]. \tag{25}$$

The prior state vector of x^- and the prior covariance of P^- are represented in Eq. (26) as

$$x^- = \begin{bmatrix} m_{10} \\ m_{01} \end{bmatrix}, \quad P^- = \begin{bmatrix} m_{20} & m_{11} \\ m_{11} & m_{02} \end{bmatrix}. \tag{26}$$

5.2 Results of the state estimation

The estimation experiment for the non-observed velocity was conducted under the displacement measurement. The true values of the displacement and the velocity were obtained by solving the equation of motion (Eq. (21)) using the Runge-Kutta method. The calculation conditions are as follows: spring constant $k = 1$; damping constant $c = 0.05$; nonlinear spring constant $\mu = 0.1$; variance of white system noise $\sigma_v^2 = 1$; and time increment for the Runge-Kutta method $\Delta t = 0.1$. The initial conditions are $x(0) = 0$ and $v(0) = 0$.
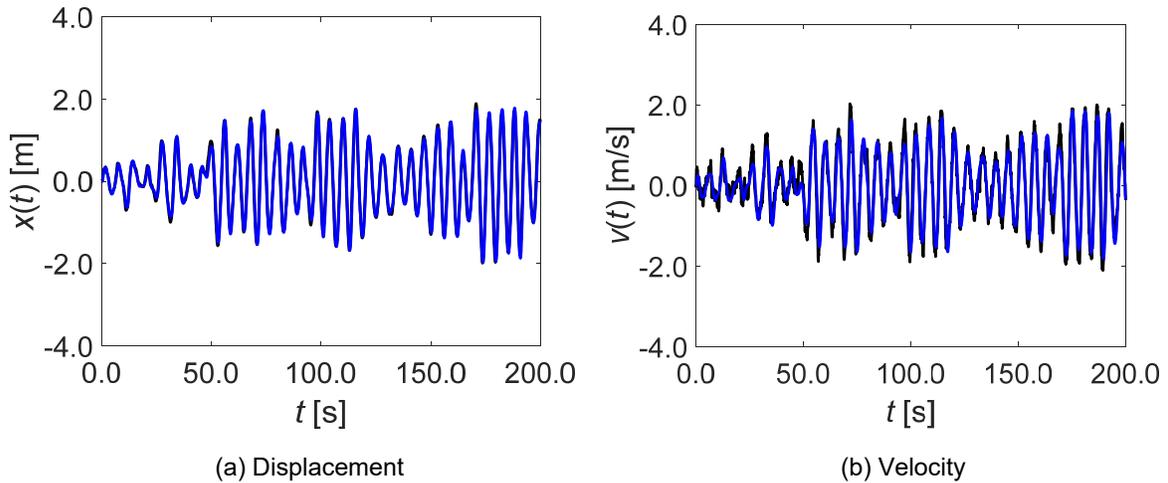

(a) Displacement

(b) Velocity

Fig. 5 State estimation results for displacement and velocity using the KF-SDA in the case of the nonlinear single-DOF system subjected to white noise excitation (the case of $\mu = 0.1$).

The true values of the displacement and the velocity are represented by the solid black line in Fig. 5. Fig. 5(a) shows the state estimation result for the displacement, while (b) shows that for the velocity. The state estimation conditions are: $R = 10^{-1}$ and $D = 0.05$. The initial conditions of each moment are as follows: $m_{10}(0) = 0$; $m_{01}(0) = 0$; $m_{20}(0) = 1$; $m_{11}(0) = 0$; and $m_{02}(0) = 1$. The estimation results of the displacement and the velocity are represented by the solid blue line in Fig. 5. Additionally, the estimation values of both displacement and velocity were in good agreement with the true values. The correlation coefficients between the true and estimated signals were 0.9813 and 0.9311 for the displacement and the velocity, respectively.

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In this case, the relationship between the restoring force $F(x)$ and the displacement x is in Fig. 6. The white noise has the variance of system noise $\sigma_v^2 = 1$. Thus, the dominant component of white noise amplitude is in ± 3 [N] ($3\sigma_v$ region i.e. contains 99.7%). Hence, the displacement response contains the area between around ± 2 [m]. Therefore, the calculation condition represents the weakly nonlinear characteristics.
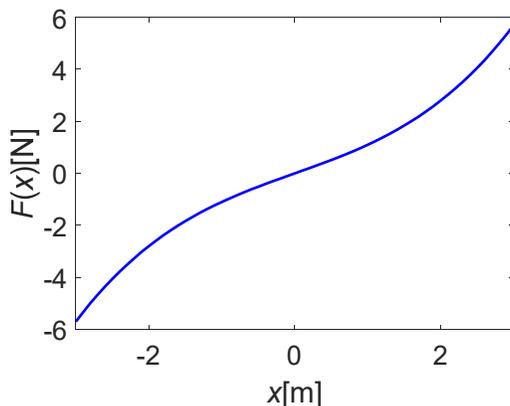

Fig. 6 Relationship between restoring force and displacement.
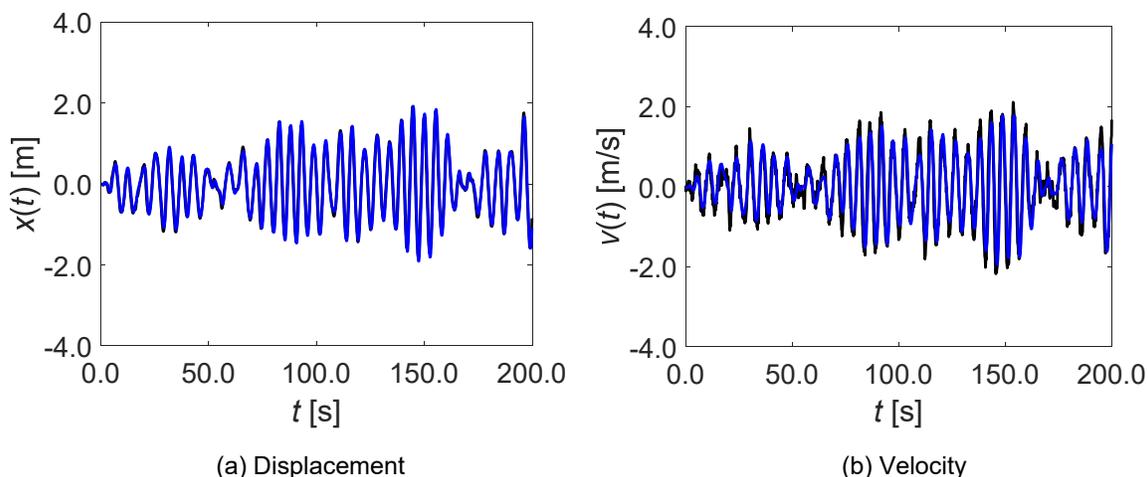

(a) Displacement

(b) Velocity

Fig. 7 State estimation results for displacement and velocity using the KF-SDA in the case of the nonlinear single-DOF system subjected to white noise excitation (the case of $\mu = 0.2$).

Moreover, the case of more strongly nonlinear ($\mu = 0.2$) is considered. The true values of the displacement and the velocity are represented by the solid black line in Fig. 7. In addition, the estimation results of the displacement and the velocity are represented by the solid blue line in Fig. 7. The correlation coefficients between the true and estimated signals were 0.9782 and 0.9218 for the displacement and the velocity, respectively. In case of the more strongly nonlinearity (i.e. more than $\mu \geq 0.2$), the correlation coefficient is decreased less than 0.9.

6. Extensibility to the filtering problem of the non-Gaussian/nonlinear system

The KF-SDA proposed in this study can be expanded to a non-Gaussian system, nonlinear system, combination system of the two systems, etc. The prediction step is required for the appropriate replacement of the moment equation described in the stochastic dynamics.

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Conversely, the dynamics of the complex probabilistic difference equation has been considered by many researchers [9,10,11]. As a classic example, the Fokker-Planck equation of the single-DOF, which was a nonlinear restoring force, was solved analytically by Caughey [12]. The analysis method of response distribution and transient moment under non-Gaussian excitation have recently been proposed in the area of continuous time stochastic dynamics [13,14,15]. These research results are expected to contribute to the evolution of the proposed filter.

7. Conclusion

This study proposed a modified Kalman filter (KF-SDA) using the solution of the moment equation in the prediction step. The following results were obtained.

(1) The modified Kalman filter was introduced based on the solution of the moment equation. In the algorithm, the prediction step was replaced by the solution of the moment equation from the solution of a linear Gaussian evolution process. Through this procedure, the prediction step can describe the unsteady/non-Gaussian/nonlinear response.

(2) The fundamental operation of the proposed Kalman filter was verified using a single-DOF system and two-DOF system subjected to white noise excitation. Subsequently, the estimation operation of the unknown state vector was confirmed.

(3) The state estimation was performed in terms of single-DOF system, which was simultaneously subjected to white noise and stochastic parametric excitations to exhibit the extensibility of the proposed modified Kalman filter. Consequently, the estimation operation of the unknown state vector was confirmed.

(4) The state estimation was performed in terms of a Duffing system, which was subjected to white noise excitation to exhibit the applicability of the proposed modified Kalman filter to a nonlinear system. Consequently, the estimation operation of the unknown state vector was confirmed.

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