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Linear system identification of 1-dof vibratory system based on the Maximum Likelihood Estimation using the analytical solution of Fokker-Planck equation

Soichiro Takata[1,a,*]

[1]Department of Mechanical Engineering, National Institute of Technology, Tokyo College, 1220-2 Kunugida-machi, Hachioji-shi, 193-0997, Japan

*Corresponding author

[a]<takata@tokyo-ct.ac.jp>

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Abstract. This paper discusses a new identification method of linear single-degree-of-freedom system using Gaussian random vibration response. The propose method is based on the method of Maximum Likelihood Estimation (MLE). The likelihood function of proposed method is composed from the analytical solution of Fokker-Planck equation. The estimation formulas of unknown parameter are obtained by maximization of the original likelihood function. The obtained estimators correspond with population variance estimation of multivariate Gaussian model. Furthermore, the numerical identifications are conducted using the random vibration response by calculation result of 4th Runge-Kutta method. In the result, the estimation performance of the propose method is confirmed in terms of dependency of sample number and dependency of damping coefficient. Especially, the proposed method is implied the application to identification problem of large damping system. Quantification of large damping characteristic is important problem, because it is very difficult problem in conventional identification method. Moreover, the benchmark tests are conducted with Half-Power Method (HPM) based on the spectral analysis and Auto-Regressive Method (ARM) based on the time series analysis, respectively. The results of benchmark are shown in the accuracy of propose method is higher than its of HPM and ARM, respectively. Finally, the expansion to recursive estimation algorithm is conducted using MLE estimator of recurrence form. In addition, operation of the recursive algorithm is confirmed.

1. Introduction

The system identification method using random vibration response is traditionally used in the field of mechanical structural vibration. The method using the auto-regressive time series analysis is the most widely known method [1,2,3,4]. Because it only needs output data and obtaining the input data is generally difficult in field test, Auto-Regressive Method (ARM) is often used in actual field data. However, because of its dependency on output data, its accuracy in variance of prediction error is low. In recent years, the ARX model which uses input-output data solves the above problem [5,6].

Operational modal analysis (OMA) is recently developing in the field of stochastic signal processing technologies [7,8,9,10]. In OMA, the mode shape visualization and the estimation of modal parameters are conducted using frequency response based on modal expansion, which uses white noise excitation as input. The modal shape in actual structure can then be understood using OMA. It is available to data analysis in actual field sensor data. However, because it is based on the frequency method, OMA cannot be applied to large damping structural and material systems, especially in resonance spectrum [11].

In this paper, we propose an identification method based on the probability density function. The shape of the probability density function obeys system parameters such as spring constant, damping constant, and diffusion coefficient of input white noise. The characteristics are unrelated to frequency spectrum and correlation function. Thus, a method based on the probability density function is expected to apply to large damping vibratory system. In linear and weakly nonlinear systems that are subjected to white random noise excitation, the analytical solution of Fokker-Planck equation can also be obtained [12,13,14]. In this study, we propose a new function using the analytical solution of Fokker-Planck equation. Moreover, the unknown parameter estimation formula has been derived using Maximum Likelihood Estimation (MLE).

2. Derivation of the Identification Algorithm

In this chapter, the proposed identification algorithms are derived based on the method of MLE. The equation of motion in single-degree-of-freedom (dof) system in case of unit mass is as follows:

$$\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} + kx(t) = w(t) \tag{1}$$

Here, $w(t)$ denotes the white noise, $N(0, \sigma_w^2)$, whereas, σ_w^2 represents the variance of input white noise. Its probabilistic differential equation is shown in the Eq. (2) with state variables, $x_1 = x(t)$ and $x_2(t) = dx_1(t)/dt$.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -kx_1 - cx_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \tag{2}$$

Using the above equations, the Fokker-Planck equation is as follows:

$$\frac{\partial f(x_1, x_2, t)}{\partial t} = -x_2 \frac{\partial f(x_1, x_2, t)}{\partial x_1} + (kx_1 + cx_2)\frac{\partial f(x_1, x_2, t)}{\partial x_2} + cf(x_1, x_2, t) + D\frac{\partial^2 f(x_1, x_2, t)}{\partial x_2^2} \tag{3}$$

Here, $f(x_1, x_2, t)$ and D represent the probability density function of the stochastic response and the diffusion coefficient, respectively.

The analytical solution of stationary Fokker-Planck equation is as follows:

$$f_s(x_1, x_2) = \frac{c\sqrt{k}}{2\pi D}\exp\left[-\frac{c}{2D}(kx_1^2 + x_2^2)\right] \tag{4}$$

Eq. (4) contains the parameters k and D/c and the stationary distribution is represented by $f_s(x_1, x_2)$.

In this paper, we will propose an estimation method of unknown parameters, k and D/c, based on MLE. Here, the likelihood function is defined by the analytical solution of the Fokker-Planck defined in Eq. (4) and is observed as time series data. The observed time series data contain the displacement, x_1, and velocity, x_2; data sets are $\mathbf{D} = \{(x_{1,1}, x_{2,1}), (x_{1,1}, x_{2,1}), \dots, (x_{1,\alpha}, x_{2,\alpha}) \dots, (x_{1,N}, x_{2,N})\}$. Therefore, the likelihood function is defined as follows:

$$L = \prod_{\alpha=1}^{N} \frac{c\sqrt{k}}{2\pi D}\exp\left[-\frac{c}{2D}(kx_{1\alpha}^2 + x_{2\alpha}^2)\right] = \left(\frac{c\sqrt{k}}{2\pi D}\right)^N \exp\left[-\frac{c}{2D}\sum_{\alpha=1}^{N}(kx_{1\alpha}^2 + x_{2\alpha}^2)\right] \tag{5}$$

Using MLE, the negative log-likelihood function is assumed as follows:

$$J = -\log L = -\log\left[\left(\frac{c\sqrt{k}}{2\pi D}\right)^{N}\right] + \frac{c}{2D}\sum_{\alpha=1}^{N}(kx_{1\alpha}^2 + x_{2\alpha}^2) \tag{6}$$

In the process of MLE, the negative log-likelihood function is partially differentiated using the unknown parameters, k and D/c. Moreover, because the case of differentiated likelihood is 0, the maximum likelihood estimators are obtained as follows:

$$k_{\mathrm{est}} = \frac{\sum_{\alpha=1}^{N} x_{2\alpha}^2}{\sum_{\alpha=1}^{N} x_{1\alpha}^2} \tag{7}$$

$$\frac{D_{\mathrm{est}}}{c_{\mathrm{est}}} = \frac{1}{N}\sum_{\alpha=1}^{N} x_{2\alpha}^2 \tag{8}$$

Eq. (7) and Eq. (8) are equal to the variances of analytical solution with respect to displacement, x_1, and velocity, x_2, in the stationary Fokker-Planck equation. Therefore, our proposed method is essentially equal to the variance estimation problem.

3. Numerical Simulation

In this chapter, we will verify operation of the proposed identification method by using the numerical consideration. In addition, the proposed method is compared with the Half Power Method (HPM) that is based on the frequency response method, and ARM that is based on the time series analysis.

3.1 Conditions

To identify the unknown system parameters, the time series is generated using the method of 4-th order Runge-Kutta. In HPM, the half spectrum is calculated by Fast Fourier Transform (FFT, there is no window function when the number of samples is 8192), associated with auto-correlation function of displacement response of 1-dof subjected to white noise excitation. Furthermore, the spring constant is estimated by the resonance frequency and the ratio between diffusion coefficient; while the damping constant is estimated by the resonance amplitude and the width of half power point in the half spectrum.

In ARM, AR coefficient and prediction error are calculated by the Yule-Walker method using the displacement response of 1-dof subjected to white noise excitation. The spring constant and damping constant are estimated by eigen frequency and damping ratio using AR coefficients. Furthermore, the diffusion coefficient is estimated using the variance of prediction error signal.

3.2 Result and Discussion

3.2.1 Dependency on the Number of Samples

The dependencies of the number of samples are shown in Fig. 1: the spring constant is $k = 1.2$; the damping constant is $c = 0.1$; variance of input white noise excitation is $\sigma_w^2 = 1$; initial conditions are $x(0) = 0$ and $v(0) = 0$; and the sampling period is $\Delta t = 0.1$. The left graph of (a) represents the estimation result of spring constant; the right graph of (b) represents the estimation result of the ratio between diffusion coefficient and damping constant. In addition, the blue, red, green, and black dotted lines show the result of the proposed method, ARM, HPM, and the true values, respectively.

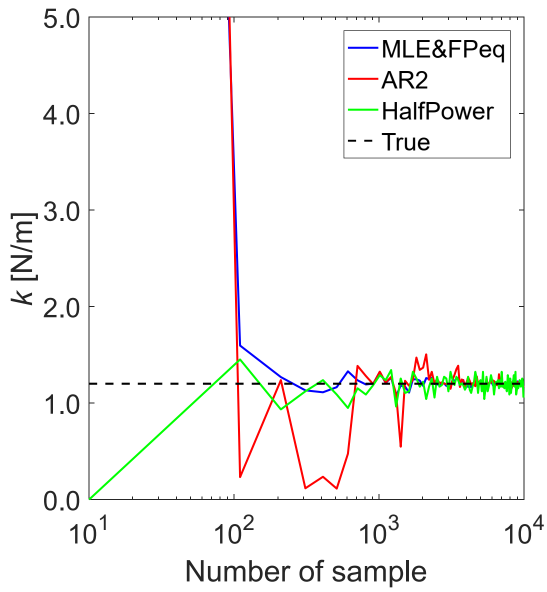
Focus on the estimation result of spring constant. The estimation values converge to the true values when the samples are at least 2000 in all estimation methods. On the other hand, the estimation values are not in agreement with the true values when there are less than 100 samples in all methods. The proposed method has the least error values which is less than 10% when the number of samples ranges from 100 to 2000.

In the case of HPM, the error between estimation and true values is approximately 20%. HPM is calculated using the FFT of the auto-correlation function. The auto-correlation function consists of the sum of the noise signal and the periodic signal of the forced vibration. Thus, the FFT of the periodic signal is independent of the noise signal. As the results showed, the decline of accuracy does not occur when the samples are low. Here, the accuracy of HPM is less than the proposed method.

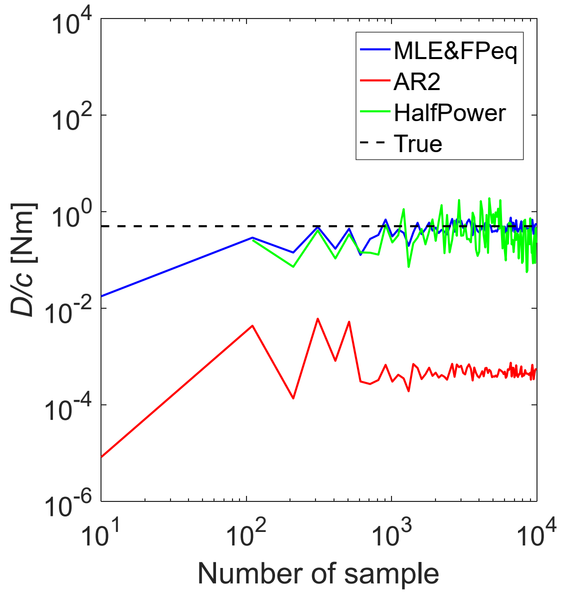
In the case of ARM, high estimation errors of more than 90% have been observed. Generally, ARM has higher accuracy of estimation performance than the FFT method in low sample conditions. However, the estimation accuracy decreased when there were less than 500 samples. In addition, the accuracy of ARM is less than the HPM and the proposed method. Therefore, the result shows the applicable limits of ARM.

Furthermore, focus on the estimation result of the ratio between the diffusion coefficient and the damping constant. The proposed method converges to the true value with increasing number of samples. The estimation errors between the estimation and true values are at the smallest in our proposed method. The proposed method has an estimation error of 30% in the range of over 1000 samples. Between 100 to 1000 samples, the estimation error is approximately 40%. Moreover, with samples less than 100, the estimation error is over 100%.

In the case of HPM, the fluctuation of estimation values increased with increasing number of samples; especially, in the case of samples less than 100, the estimation value is not obtained in order to estimate the resonance frequency is 0. On the other hand, in the case of ARM, errors between estimation and true values occurred in the 10^3 number of samples.



(a) the case of spring constant



(b) the case of ratio between diffusion coefficient and damping coefficient

Fig. 1. Benchmark test results in terms of sample number dependency.

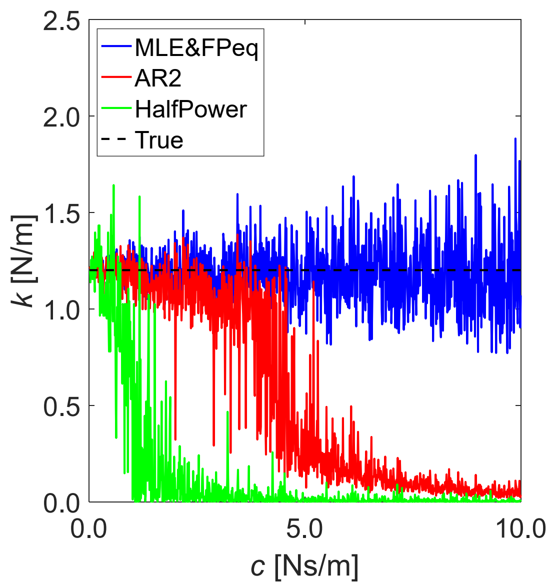
3.2.2 Dependency on the Damping Coefficients

The dependencies of the damping constant are shown in Fig. 2: the spring constant is $k = 1.2$; variance of input white noise excitation is $\sigma_w^2 = 1$; initial conditions are $x(0) = 0$ and $v(0) = 0$; sampling period is $\Delta t = 0.1$.; and the damping constants are in the range of $c = 0.01$ to $c = 10$. The left graph of (a) represents the estimation result of spring constant while the right one of (b) represents the estimation result of the ratio between diffusion coefficient and damping constant. In addition, the blue, red, green, and black dotted lines show the result of proposed method, ARM, HPM, and the true values, respectively.

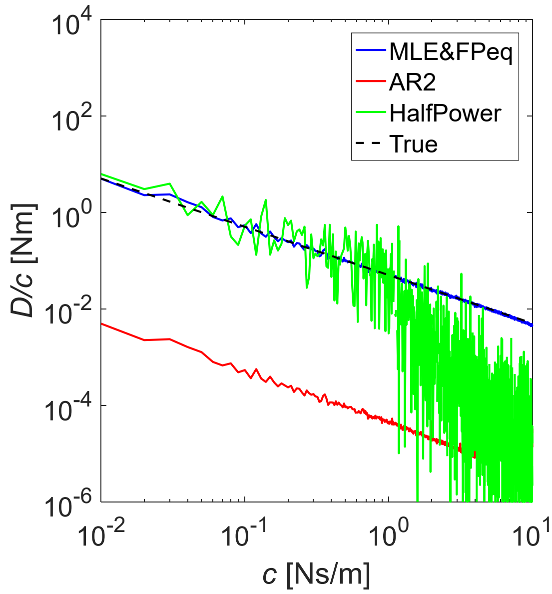
Here, the focus is on the estimation result of spring constant (in case of (a)). Our proposed method is in good agreement with the true values throughout the range of values of the damping constant. On the other hand, HPM and ARM are not in agreement with the true values when the damping constant is between 0.5 and 1.8. Our proposed method is based on the variance ratio of probability density function between displacement and velocity. The probability density functions are obtained both in low signal and high noise levels. As a result, the variance ratio of probability density function between displacement and velocity can be precisely calculated. The proposed method estimates using the raw time series data and is independent of the auto-correlation function and frequency spectrum data.

On the other hand, HPM and ARM are based on frequency and auto-correlation function domains. In these methods, the noise component is more dominant than the signal. As a result, the spring constant evaluated is less than true values.

Furthermore, focus on the estimation result of the ratio between the diffusion coefficient and the damping constant (in case of (b)). The proposed method is in good agreement with the true values throughout the entire range of ratio between diffusion coefficient and damping constant. HPM is not in agreement with the true values when $c > 0.8$. ARM is not in agreement with the true values throughout the entire range of the D/c parameter.



(a) the case of spring constant



(b) the case of ratio between diffusion coefficient and damping coefficient

Fig. 2. Benchmark test result in terms of damping coefficient dependency.

4. Extension to the Recursive Algorithm

This chapter discusses the expansion of the above proposed method to the recursive algorithm to save the capacity of the memory. The recurrence formula is derived by the relationship of the root mean square as follows:

$$k_{\text{est}}[j] = \frac{f[j]}{g[j]} \tag{9}$$

$$f[j] = \frac{j-1}{j} f[j-1] + \frac{1}{j} x_2^2[j] \tag{10}$$

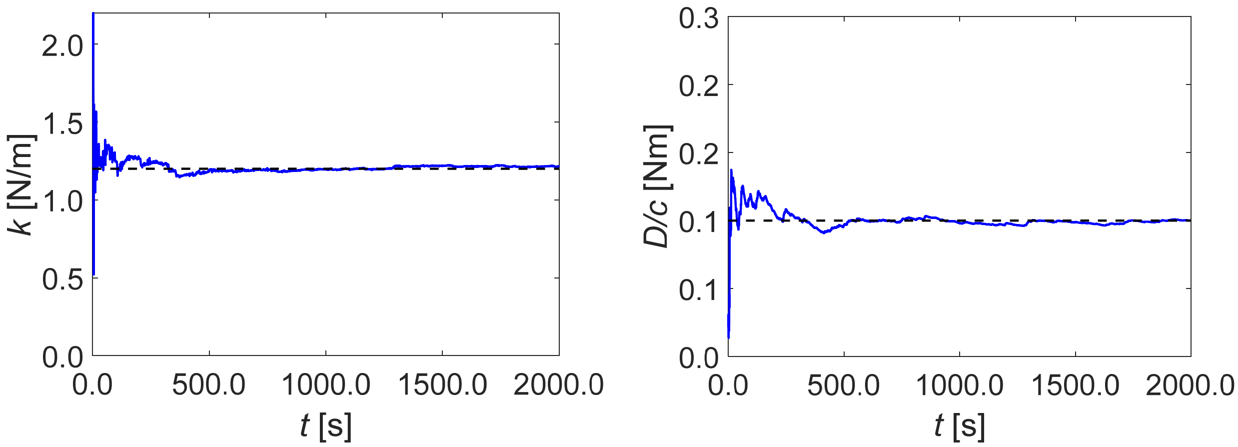
$$g[j] = \frac{j-1}{j} g[j-1] + \frac{1}{j} x_1^2[j] \tag{11}$$

Here, the initial values of $f[0]$ and $g[0]$ are zeros. In addition, variable j represents an integer to distinguish it from the continuous time variable, t. The ratio between the diffusion coefficient and the damping constant is as follows:

$$\frac{D_{\mathrm{est}}}{c_{\mathrm{est}}}[j] = \frac{j-1}{j} \frac{D_{\mathrm{est}}}{c_{\mathrm{est}}}[j-1] + \frac{1}{j} x_2^2[j] \tag{12}$$

The initial value is obtained as a result of the above batch processing algorithm. The estimation results using Eq. (9) ~ (12) are shown in Fig. 3. Here, the horizontal axes represent the time; the vertical axes represent the spring constant (a) and the ratio between the diffusion coefficient and the damping constant (b). The blue solid and black dotted lines show the estimation values of the unknown parameter and true values, respectively. Here, the spring constant is $k = 1.2$; the damping constant is $c = 0.5$; variance of input white noise excitation is $\sigma_w^2 = 1$; initial conditions are $x(0) = 0$ and $v(0) = 0$; and the sampling period is $\Delta t = 0.1$.

The estimation results of both the spring constant and the ratio between the diffusion coefficient and damping constant converge to the true values. Therefore, the operation of recursive algorithm is verified. With the proposed online estimation algorithm, *t* should be greater than 500 to converge with true values, with the number of samples being 5000. On the other hand, 1000 samples are necessary for the above batch processing. Therefore, the number of samples for the proposed recursive algorithm is approximately the same as number of samples of the above batch algorithm.



(a) the case of spring constant

(b) the case of ratio between diffusion coefficient and damping coefficient

Fig. 3. Estimation result using the proposed online estimation algorithm.

5. Damping Coefficients Dependency on Recursive Algorithm

In this chapter, the damping coefficient dependency of the recursive algorithm is considered. Here, the spring constant is $k = 1.2$; variance of input white noise excitation is $\sigma_w^2 = 1$; initial conditions are $x(0) = 0$ and $v(0) = 0$; sampling period is $\Delta t = 0.1$; and the damping constants are between $c = 0.1$ to $c = 10$. The recursive algorithm is performed till 2000 s; then, the ratio between the true and estimation value is calculated by the converged estimation value using the proposed algorithm. The estimation results are shown in Fig. 4 and 5. In Fig. 4, the horizontal axis represents the damping coefficient while the vertical axis represents the ratio between true value and estimation value. The centers of the ratio is near the value of 1 and fluctuates in the 0.8 ~ 1.2 range. The operation of the proposed algorithm to estimate the spring constant is confirmed in the large damping system. Furthermore, focus on the estimation result of ratio between the diffusion coefficient and the damping constant in Fig. 5. The ratio between the diffusion coefficient and the damping constant decreases as the damping constant increases and slightly fluctuates between 0.9 and 1.05. The operation of the proposed algorithm to estimate the ratio between the diffusion coefficient and the damping constant is confirmed in the large damping system.

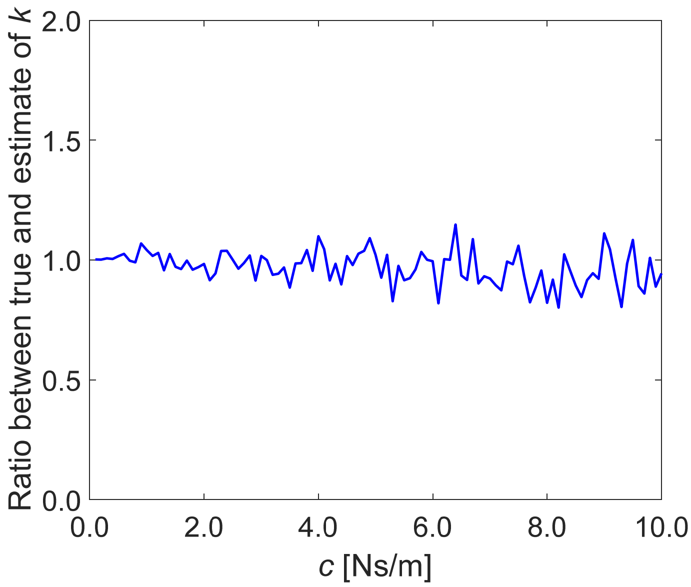


Fig. 4. Damping coefficient dependency on estimation accuracy in case of spring constant.

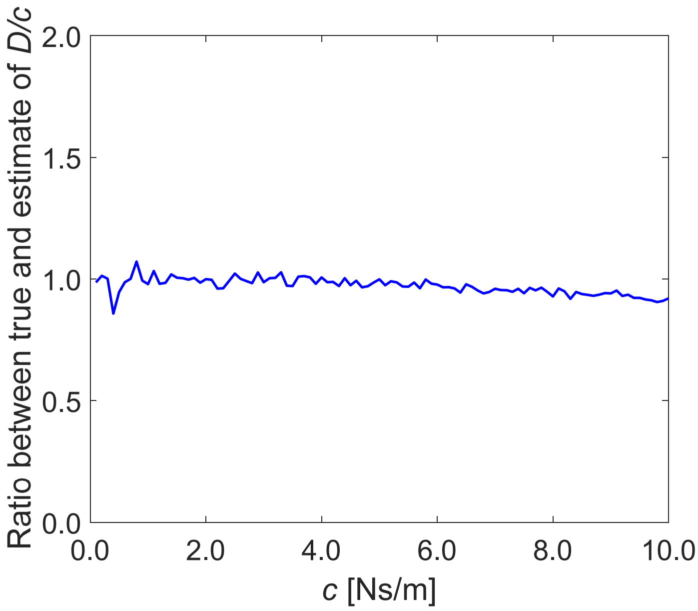


Fig. 5. Damping coefficient dependency on estimation accuracy in case of ratio between diffusion coefficient and damping coefficient.

6. Conclusions

In this paper, we discussed the identification method based on the MLE using analytical solution of Fokker-Planck equation. As the result, the following results are obtained.

(1) Estimation formulas of the spring constant and the ratio between the diffusion coefficient and the damping constant were derived using MLE method using Analytical solution of Fokker-Planck equation.

(2) The benchmark tests in terms of estimation accuracy were conducted in cases of HPM and ARM, respectively. As the result, our proposed method showed the most accuracy in cases of the number of samples dependence and damping constant dependence.

(3) The expansion to the recursive algorithm were conducted beyond the batch processing algorithm. In addition, the estimation experiments were conducted using the numerical simulation.

(4) The verification of proposed method in terms of estimation accuracy were conducted. As the result, our proposed method showed the applicability to large damping system.

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