Journal of Mechanical and Electrical Intelligent System (JMEIS)

Mathematical Model and Analysis of 4-Stage Passive RC Polyphase Filter for Low-IF Receiver

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Keywords: superposition, complex signals, polyphase filter, low-IF receiver, image rejection.

Abstract. The objective of this paper is to introduce a mathematical model of a 4-stage RC polyphase filter for a low intermediate frequency receiver in wireless communication systems. The transfer function of this filter is simply derived based on the widened superposition principle. In addition, the image rejection ratio of this network is also mathematically derived. At the baseband, the phase sequence orders of the wanted signals and the image signals are conflicted. The concepts of negative and positive Hilbert transform pairs divide the received signals into negative and positive polyphase signals. We express these signals on the angular frequency plane and plot the transfer function of this filter on both negative and positive frequency domains. The characteristics of the transfer function are mathematically analyzed and plotted on Matlab and SPICE. There is a tradeoff among signal to noise ratio, power loss and chip area. The values of resistors and capacitors are chosen based on the physical sizes of these components. The filter works at the 5 MHz center frequency and provides 36 dB image rejection ratio for a 10 MHz bandwidth receiver.

1. Introduction

Complex signal processing is a fast growing area today and a desired effectiveness in utilization of bandwidth and energy makes the progress even faster [1]. Moreover, the demand for fully integrated Radio Frequency (RF) transceiver in wireless communication gives rise to great attention to single chip low power transceivers [2]. The analog section of the low intermediate frequency (low-IF) receiver includes a polyphase filter, a programmable gain amplifier, and an analog to digital converter. All components in this analog section can be integrated in one chip. Therefore, polyphase filters are widely used in these RF front-end systems [3]. In this research, we propose a new technique to derive the transfer function and a mathematical model of a 4-stage passive RC polyphase filter for a low-IF receiver.

The main contribution of this research comes from the mathematical model of a passive polyphase filter which is derived based on widened superposition principle. There are 7 sections in this paper. Section 2 constitutes background knowledge, with an explanation of the necessity for network analysis based on the widened superposition principle. Section 3 mathematically analyzes illustrative complex signals considered in details. Section 4 focuses on the frequency domain analysis of the 4-stage RC polyphase filter. SPICE simulation results for the proposed design of the polyphase filter are described in Section 5. A brief discussion of the research results is given in Section 6. The main points of this work are summarized in Section 7.

2. Design considerations for polyphase filter

2.1. Widened superposition principle

In this section, we propose a new concept of the superposition principle which is useful for deriving the transfer function of a network. The conventional superposition theorem is used to find the solution to linear networks consisting of two or more sources (independent sources, linear dependent sources) that are not in series or parallel. To consider the effects of each source independently requires that sources be removed and replaced without affecting the final result. Therefore, to remove a voltage source when applying this theorem, the difference in potential between the terminals of the voltage source must be set to zero (short circuit); removing a current source requires that its terminals be opened (open circuit). This procedure is followed for each source in turn, and then the resultant responses are added to determine the true operation of the circuit.

There are some limitations of conventional superposition theorem. Superposition cannot be applied to power effects because the power is related to the square of the voltage across a resistor or the current through a resistor. Superposition theorem cannot be applied for non-linear circuit (diodes or transistors). In order to calculate the load current or the load voltage for the several choices of the load resistance of the resistive network, one needs to solve for every source voltage and current, perhaps several times. With the simple circuit, this is fairly easy but in a large circuit this method becomes a painful experience.

In this paper, the nodal analysis on circuits is used to obtain multiple Kirchhoff current law equations. The term of "widened superposition" is proposed to define a general superposition principle which is the standard nodal analysis equation and simplified for the case when the impedance from node A to ground is infinity and the current injection into node A is 0. In a circuit having more than one independent source, we can consider the effects of all the sources at a time. The widened superposition principle is used to derive the transfer function of a network [4, 5, 6]. Energy at one place is proportional to their input sources and the resistance distances of transmission spaces. Let $E_A(t)$ be energy at one place of multi-sources $E_i(t)$ which are transmitted on the different resistance distances d_i (R, Z_L, and Z_C in electronic circuits) of the transmission spaces as shown in Fig. 1.

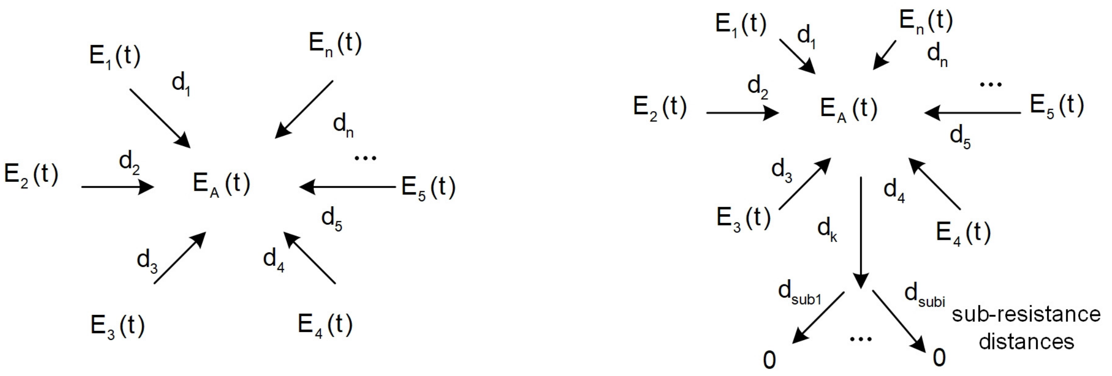
The widened superposition principle is defined as

$$E_A(t)\sum_{i=1}^{n}\frac{1}{d_i}=\sum_{i=1}^{n}\frac{E_i(t)}{d_i} \tag{1}$$

If the transmission spaces are divided into small branches, energy at every small place will be calculated by the direct resistance distances of every branch. Here, d_{subi} are sub-resistance distances. In this case, the resistance distance of each node is calculated from the input loop to the output loop.

$$E_A(t)\left(\sum_{i=1}^{n}\frac{1}{d_i}+\cfrac{1}{d_k+\cfrac{1}{\cfrac{1}{d_{sub1}}+\cfrac{1}{d_{sub2}}+\ldots}}\right)=\sum_{i=1}^{n}\frac{E_i(t)}{d_i} \tag{2}$$

The import of these concepts into circuit theory is relatively new with much recent progress regarding filter theory, analysis and implementation.

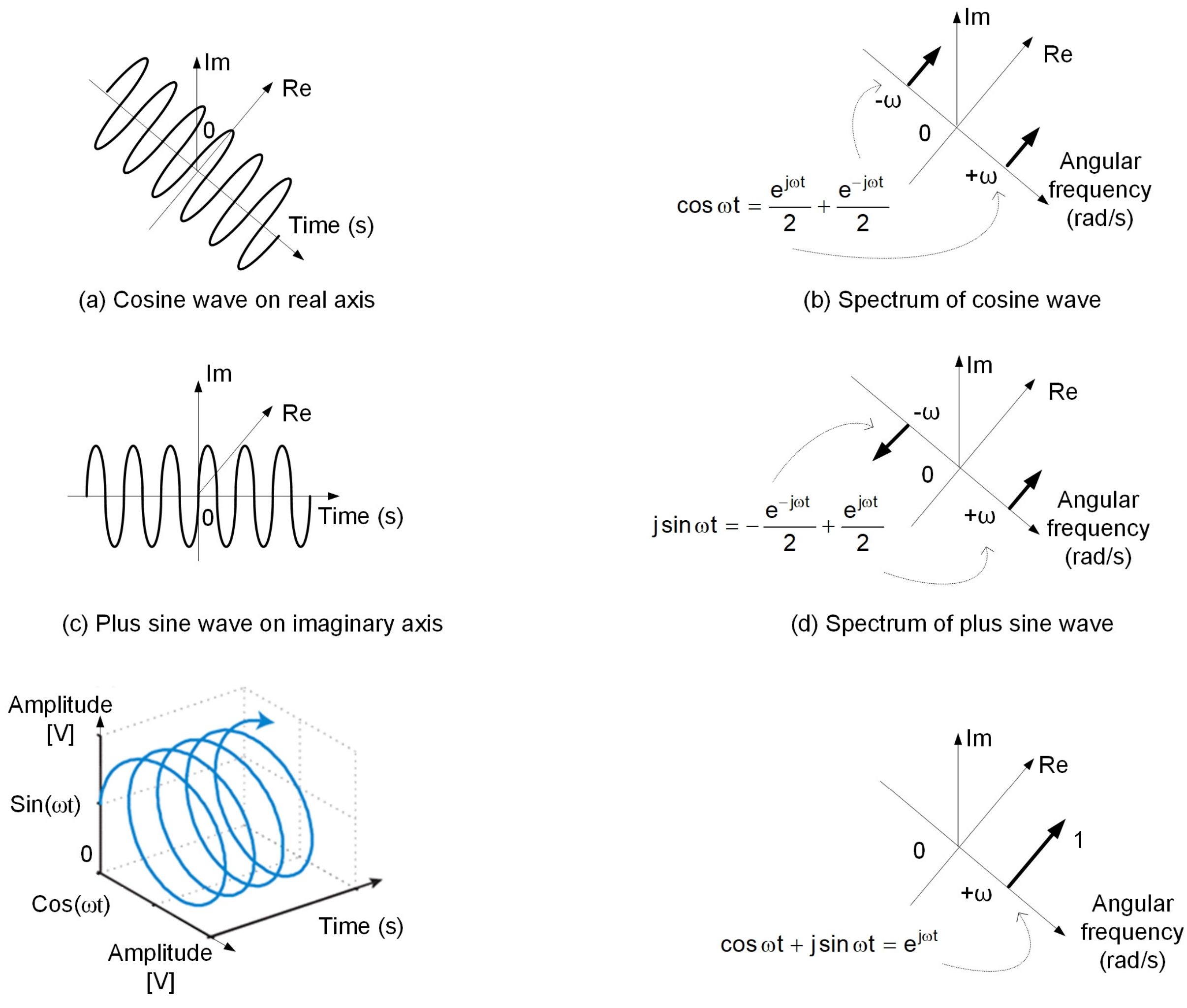


(a) Direct transmission spaces (b) Transmission spaces with sub-branches

Fig. 1. Energy at one node based on widened superposition principle and sub-branches.

2.2. Direction of wave propagation

This section introduces a new concept of phase-order in a real signal. The spectrum analysis is the single most important topic in the field of analog signal processing and in many other fields. The best way to understand the true nature of a real-world signal is by examining its spectrum to make the difficult concepts clear and easy to grasp.



(a) Cosine wave on real axis

(b) Spectrum of cosine wave

(c) Plus sine wave on imaginary axis

(d) Spectrum of plus sine wave

(e) Positive angular frequency wave

(f) Spectrum of positive angular frequency wave

Fig. 2. Positive angular frequency wave and spectrum.

Moreover, frequency is the number of occurrences of a repeating event per unit of time. It is also referred to as temporal frequency, which emphasizes the contrast to angular frequency. A periodic signal of any shape can be described using the Fourier series which contains sine and cosine expressions. A much more simplified method to write the Fourier series is based on the well-known Euler form for sinusoidal signals. This method to write the Fourier series in complex form was the background to talk about positive and negative frequencies. However, to circumvent these terms, which might cause problems to understand the meaning of negative frequencies, we can avoid the word "frequency" in this context. In this paper, the sign of the angular frequency (ω) is proposed to represent the direction of wave propagation.

Base on the Euler's equation $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$, the angular frequency of the complex exponential function $e^{j\omega t}$ is positive as shown in Fig. 2. Likewise, $e^{-j\omega t}$ is a conjugate of Euler's equation, where the angular frequency of $e^{-j\omega t}$ is negative as shown in Fig. 3. Of course, both angular frequencies rotate in different directions, however, both with the same frequency which still is a pure number. Negative angular frequency represents a clockwise traveling wave, while positive angular frequency represents an anti-clockwise traveling wave.

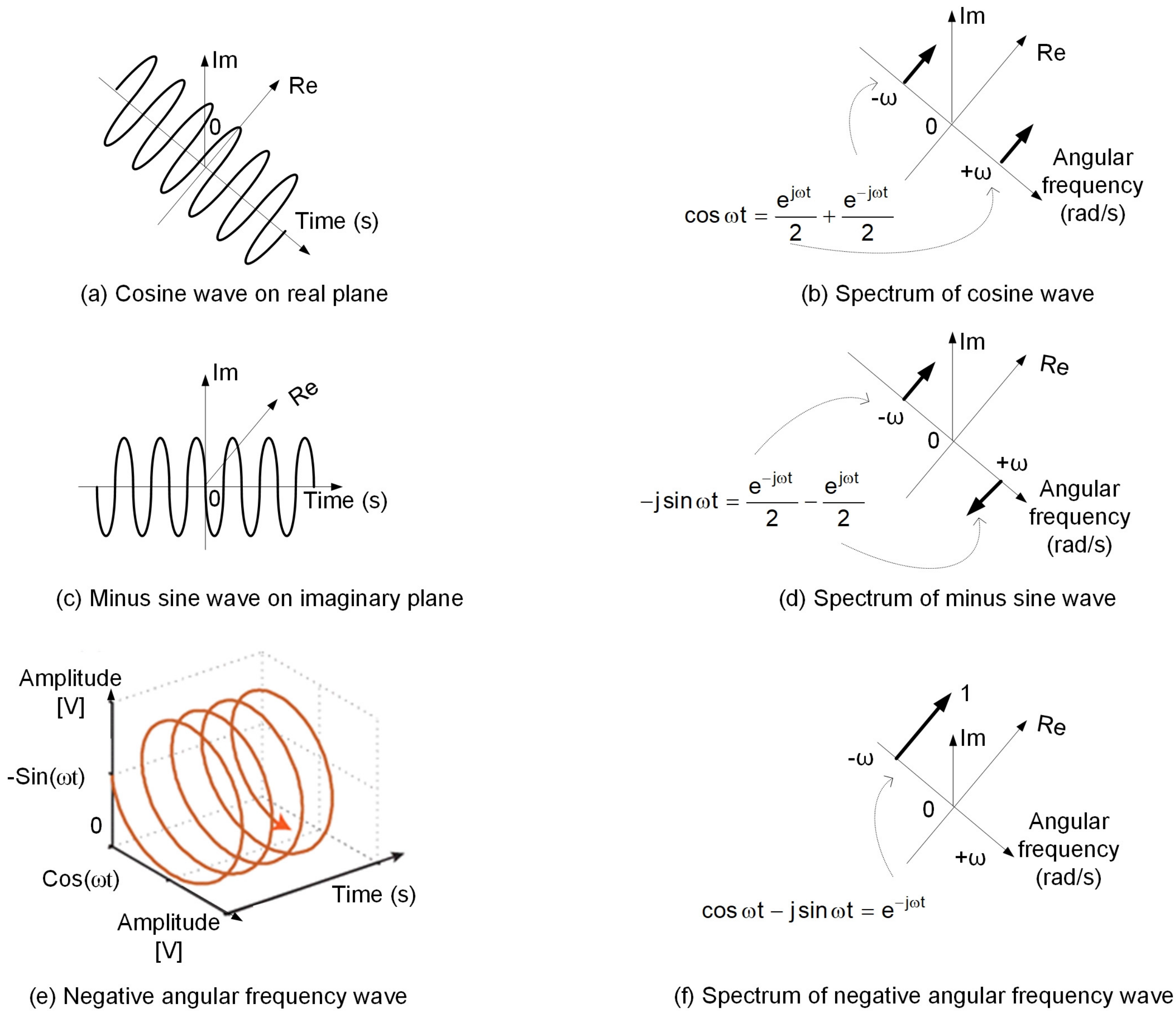


Fig. 3. Negative angular frequency wave and spectrum.

2.3. Hilbert transform pairs

This section recalls some Hilbert transform pairs in analog complex signal processing. In the beginning of the 20th century, the German scientist David Hilbert showed that the function sin(ωt) is the Hilbert transform of cos(ωt). This gave us the ±π/2 phase-shift operator which is a basic property of the Hilbert transform. Therefore, the relationship between the real part and the imaginary part of a complex signal is generally described by a Hilbert transform. Hilbert transformer is also called a quadratic filter in many analog circuits.

The Hilbert transform $H\{x(t)\}$ of $x(t)$ is given by:

$$H\{x(t)\} = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{x(t-\tau)}{\tau} d\tau \tag{4}$$

where, *"PV"* is the *Cauchy Principal Value* of the integral [7].Some properties of the Hilbert transform are summarized on Table 1.

Hilbert transforms do not change the magnitudes of signals; the phases of signals are shifted. Negative Hilbert transform shifts a *-π/2* of phase and *+π/2* of phase in the positive case.

Table 1: Hilbert transform pairs

Name	Function	Positive Hilbert Transform (ω > 0)	Negative Hilbert Transform (ω < 0)
Sine	sin(ωt)	-cos(ωt)	cos(ωt)
Cosine	cos(ωt)	sin(ωt)	-sin(ωt)
Exponential	$e^{j\omega t}$	$-je^{j\omega t}$	$je^{j\omega t}$
	$e^{-j\omega t}$	$je^{-j\omega t}$	$-je^{-j\omega t}$

3. Analysis of complex signals

3.1. Quadratic filter

In this section, the low pass filters and the complementary high pass filters (HPF) are used to constitute a quadratic filter. The in-phase and quadratic phase signals are generated based on a set of a low-pass and a complementary high-pass [8]. On the other hand, the quadratic filter is also used to reject the image signal in low-IF receivers as shown in Fig. 4. In case of differential quadratic signals, this filter is also called polyphase filter.

3.2. Received signals in low-IF receiver

The phase-orders of the received signals in a low-IF receiver are presented in this section. A polyphase filter is used to reject the image noise in a low-IF receiver as shown in Fig. 5. After being converted into the low-IF, the wanted and the image signals are located at the same frequency [9]. However, the direction of the unwanted and the desired signals are conflicted or two sets of differential signals move on the same road and kill each other. On the frequency plane, it is difficult to express the direction of these signals as shown in Fig. 6(a). On the phase-order plane, the phase-orders of the received signals are clockwise and anti-clockwise as shown in Figs. 6(b) and 6(c).

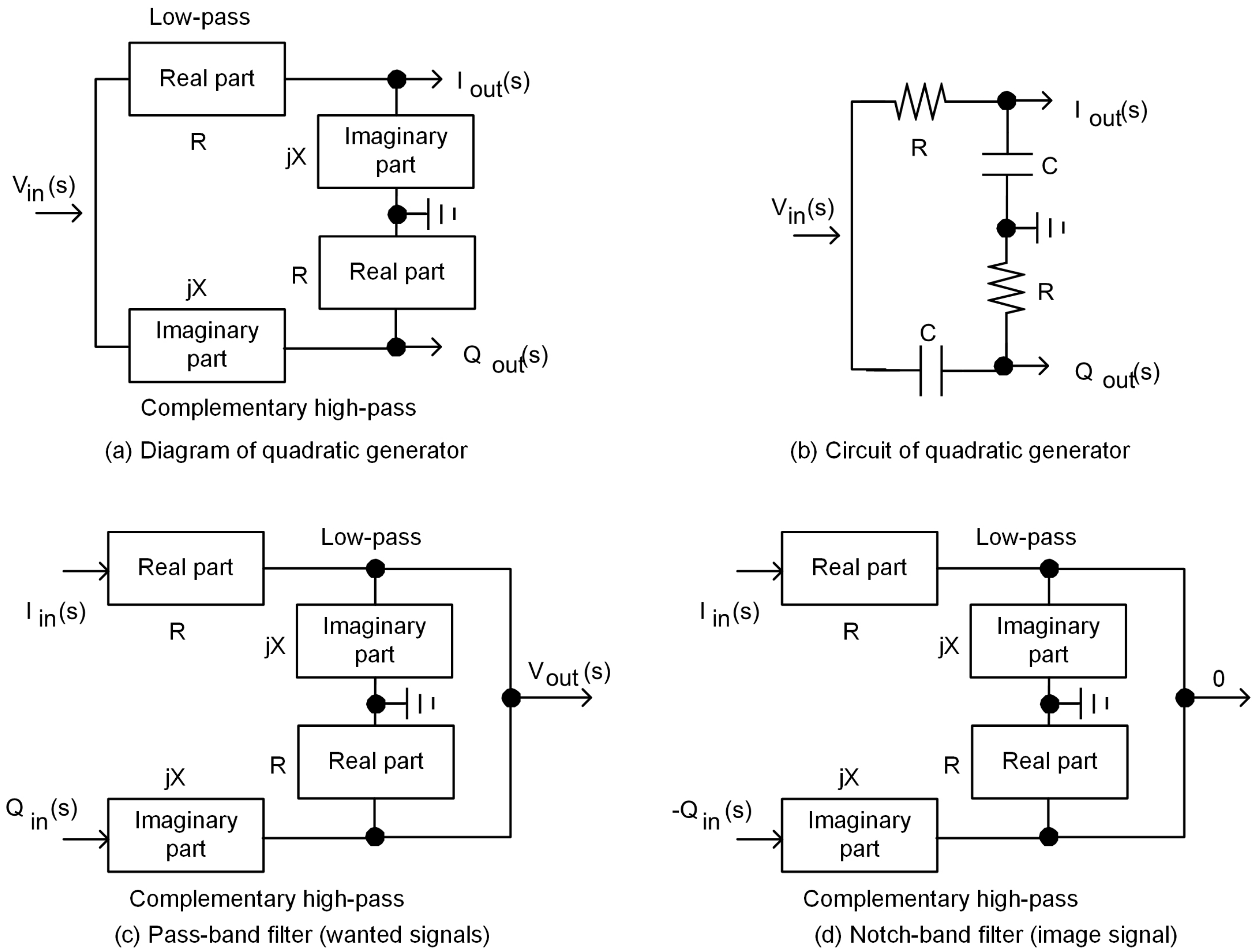


Fig. 4. Quadratic filter based on low-pass and complementary high-pass.

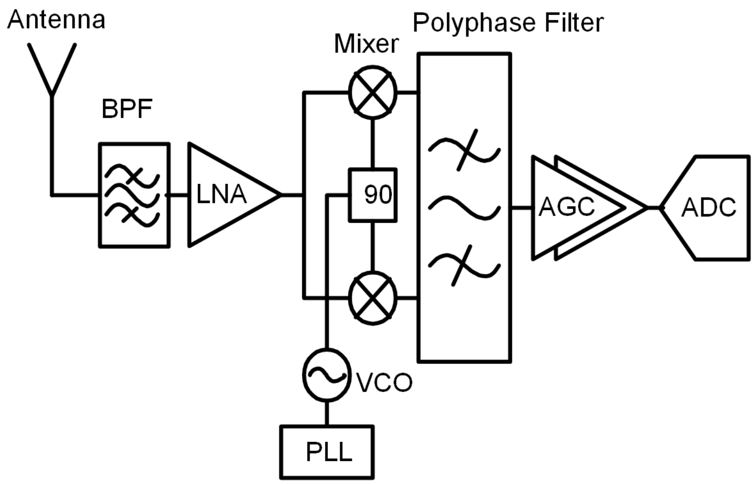
3.3. Polyphase signals and angular frequency plane

This section describes a definition of a complex signal in a low-IF receiver. The concepts of clockwise and anti-clockwise phase-orders are used to define the direction of multi-sources which are compared to the normal space. A multi-phase signal is a set of multi-sources with the same frequency and their phases are separated by the same degree.

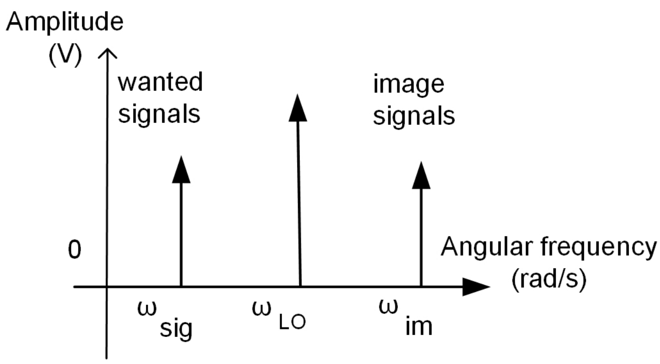
The phases of a complex signal (two sources) and a polyphase signal (four sources) are separated by 90 degrees. Moreover, a polyphase signal is also called as a complex signal. Therefore, a polyphase filter is also called as a complex filter [10]. The notations of +j, and -j are used to denote the directions of two sources are different by 90 degrees.

The positive and negative polyphase signals are expressed by two sets of phasors on the phase-order planes. There are four sources in the received signals which are called polyphase signals. Based on the properties of the Hilbert transform pairs, the clockwise phase-order of the polyphase signal is a negative frequency signal and vice versa, as shown in Figs. 7(a) and 7(b).

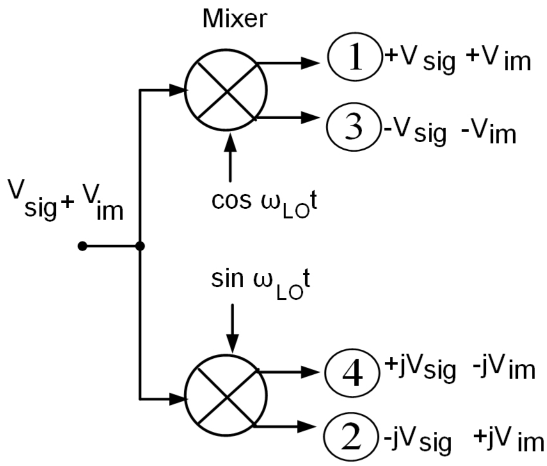
The concepts of positive and negative angular frequencies are used to define the phase directions of down-conversion signals [11].The angular frequency variables ω and $-\omega$ are proposed to represent these received signals on the angular frequency plane as described in Fig. 7(c).



(a) Architecture of Low-IF receiver

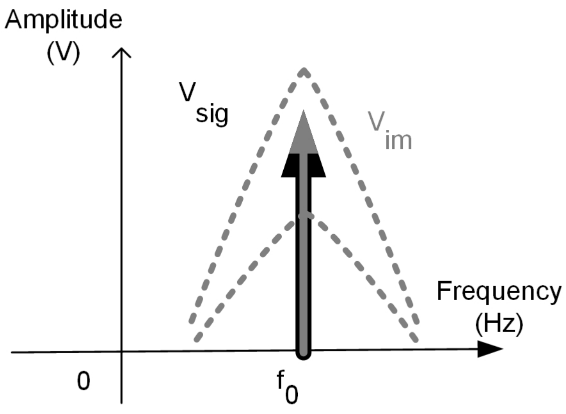


(b) Spectrum of RF signals

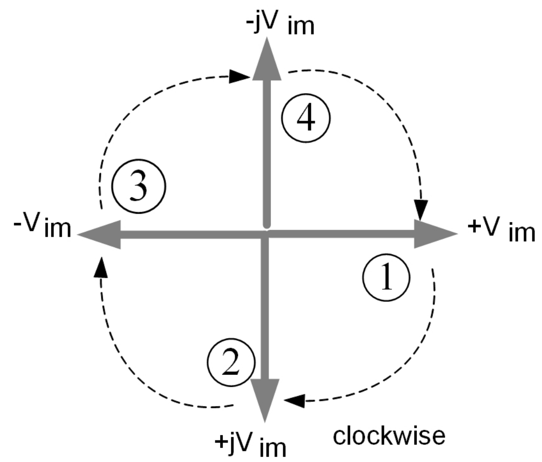


(c) Step-down conversion signals

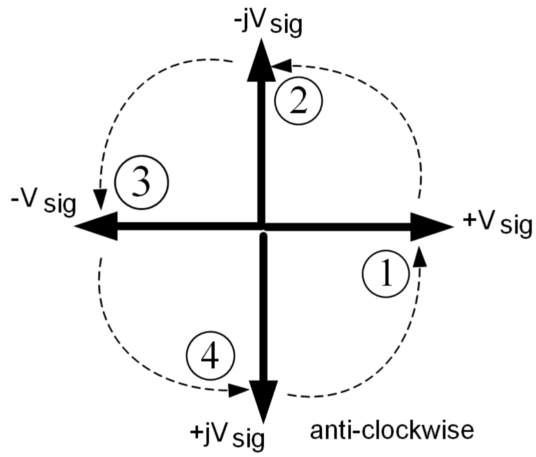
Fig. 5. Low-IF receiver and received signals.



(a) Frequency plane of received signals



(b) Phase-order of image signals



(c) Phase-order of wanted signals

Fig. 6. Frequency and phase-order planes of received signals.

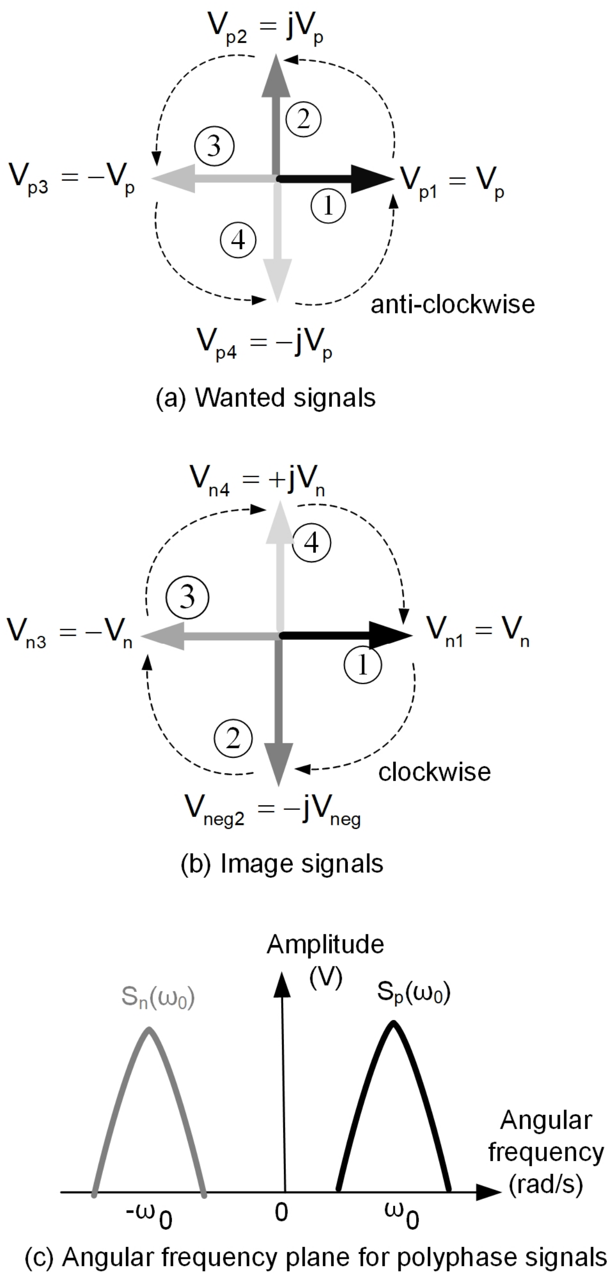


Fig. 7. Polyphase signals and angular frequency plane of received signals.

Let $V_p(t) = A\cos(\omega t + \theta_p)$ be a normal voltage, a positive polyphase signal can be defined as

$$S_P\{V_{p1}(t);V_{p2}(t);V_{p3}(t);V_{p4}(t)\} = \{1; j; -1; -j\}V_p(t) = \left\{1; e^{j\frac{\pi}{2}}; e^{j\pi}; e^{j\frac{3\pi}{2}}\right\}A\cos(\omega t + \theta_p) \tag{5}$$

Let $V_n(t) = B\cos(\omega t + \theta_n)$ be a normal voltage, a negative polyphase signal can be defined as:

$$S_n\{V_{n1}(t);V_{n2}(t);V_{n3}(t);V_{n4}(t)\} = \{1; -j; -1; +j\}V_n(t) = \left\{1; e^{-j\frac{\pi}{2}}; e^{-j\pi}; e^{-j\frac{3\pi}{2}}\right\}B\cos(\omega t + \theta_n) \tag{6}$$

Furthermore, band polyphase signals are multi-sources which include many closed frequencies. Let $V_{pb}(t)=\sum_{k=1}^{n}A_k\cos(\omega_k t+\theta_{pk})$ be a normal band voltage, a positive band polyphase signal can be defined as

$$S_{Pb}\{V_{pb1}(t);V_{pb2}(t);V_{pb3}(t);V_{pb4}(t)\}=\{1;j;-1;-j\}V_{pb}(t)=\left\{1;e^{j\frac{\pi}{2}};e^{j\pi};e^{j\frac{3\pi}{2}}\right\}\sum_{k=1}^{n}A_k\cos(\omega_k t+\theta_{pk}) \tag{7}$$

The phase relationship of these positive band polyphase signals is expressed as a negative j.

$$\frac{V_{p1,2,3,4}(t)}{V_{p2,3,4,1}(t)}=\frac{V_{p1}(t)}{V_{p2}(t)}=\frac{V_{p2}(t)}{V_{p3}(t)}=\frac{V_{p3}(t)}{V_{p4}(t)}=\frac{V_{p4}(t)}{V_{p1}(t)}=-j \tag{8}$$

Let $V_{nb}(t)=\sum_{k=1}^{n}B_k\cos(\omega_k t+\theta_{nk})$ be a normal band voltage, a negative band polyphase signal can be defined as

$$S_N\{V_{n1}(t);V_{n2}(t);V_{n3}(t);V_{n4}(t)\}=\{1;-j;-1;+j\}V_{nb}(t)=\left\{1;e^{-j\frac{\pi}{2}};e^{-j\pi};e^{-j\frac{3\pi}{2}}\right\}\sum_{k=1}^{n}B_k\cos(\omega_k t+\theta_{nk}) \tag{9}$$

Therefore, the phase relationship is expressed as a positive j.

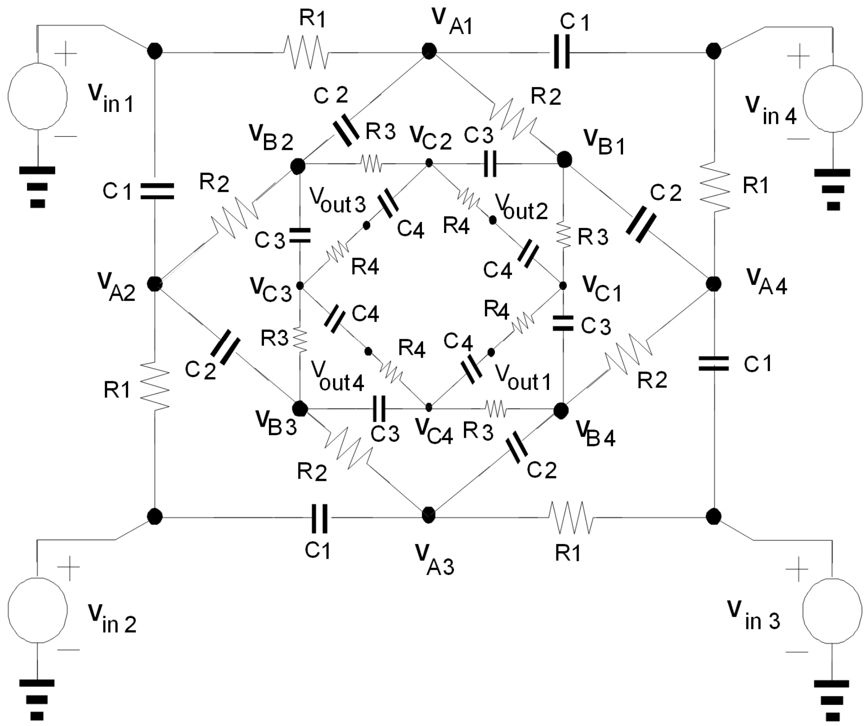
$$\frac{V_{n1,2,3,4}(t)}{V_{n2,3,4,1}(t)}=\frac{V_{n1}(t)}{V_{n2}(t)}=\frac{V_{n2}(t)}{V_{n3}(t)}=\frac{V_{n3}(t)}{V_{n4}(t)}=\frac{V_{n4}(t)}{V_{n1}(t)}=+j \tag{10}$$

4. Analysis of 4-stage RC polyphase network

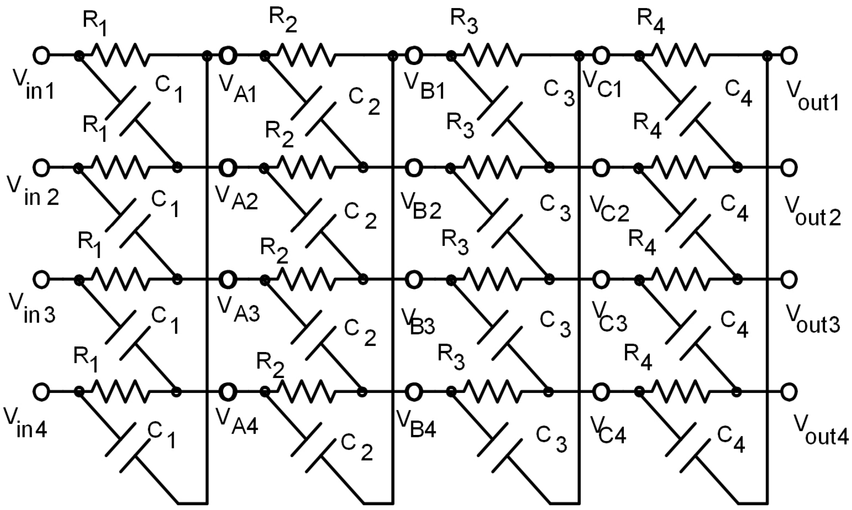
4.1. Output voltages of 4-stage RC polyphase network

This section describes a derivation of the output voltages of a 4-stage RC polyphase filter. Fig. 8 shows the loop form and the chain form of a 4-stage RC polyphase filter. The bandwidth of this system is 10 MHz and the center frequency of the baseband is 5 MHz. The image rejection ratio of this system is required greater than 30 dB. On this model, the parasitic of RC components and the time delay of the signal propagation are neglected.

Assume that $\omega=2\pi f=2\pi/T$ is the angular frequency, and the impedances of the capacitor C1 to C4 are defined as Z_{C1} to Z_{C4}. On the sub-branches, the resistance distance of each node is calculated from the input loop to the output loop. The input voltages $V_{in1,2,3,4}$ are the notations of 4 input sources from V_{in1} to V_{in4}. And, $V_{A1,2,3,4}$, $V_{B1,2,3,4}$ $V_{C1,2,3,4}$, and $V_{out1,2,3,4}$ are the voltages at every node in the circuit.



(a) Loop form of 4-stage RC polyphase ciruit



(b) Chain form of 4-stage RC polyphase ciruit

Fig. 8. Loop form and chain form of 4-stage RC polyphase filter.

Let us employ the superposition principle at each loop, and the voltages are calculated by the following nodal equations

$$V_{A1,2,3,4}\,Y_A = \frac{V_{in1,2,3,4}}{R_1} + \frac{V_{in4,1,2,3}}{Z_{C1}};\; V_{B1,2,3,4}\,Y_B = \frac{V_{A1,2,3,4}}{R_2} + \frac{V_{A4,1,2,3}}{Z_{C2}}$$
$$V_{C1,2,3,4}\,Y_C = \frac{V_{B1,2,3,4}}{R_3} + \frac{V_{B4,1,2,3}}{Z_{C3}};\; V_{out1,2,3,4}\,Y_D = \frac{V_{C1,2,3,4}}{R_4} + \frac{V_{C4,1,2,3}}{Z_{C4}} \quad (11)$$

Here, $Y_A ; Y_B ; Y_C ; Y_D$ are admittances seen looking into respective node excluding R_k and Z_{Ck} (k=1,2,3,4).

$$
\begin{aligned}
Y_A &= \frac{1}{R_1} + \frac{1}{Z_{C1}} + \frac{1}{R_2 + \dfrac{1}{\dfrac{1}{Z_{C2}} + \dfrac{1}{R_3 + \dfrac{1}{\dfrac{1}{Z_{C3}} + \dfrac{2}{Z_{C4}+R_4}}} + \dfrac{1}{Z_{C3} + \dfrac{1}{\dfrac{1}{R_3} + \dfrac{2}{Z_{C4}+R_4}}}}} + \frac{1}{Z_{C2} + \dfrac{1}{\dfrac{1}{R_2} + \dfrac{1}{R_3 + \dfrac{1}{\dfrac{1}{Z_{C3}} + \dfrac{2}{Z_{C4}+R_4}}} + \dfrac{1}{Z_{C3} + \dfrac{1}{\dfrac{1}{R_3} + \dfrac{2}{Z_{C4}+R_4}}}}} \\
Y_B &= \frac{1}{R_2} + \frac{1}{Z_{C2}} + \frac{1}{R_3 + \dfrac{1}{\dfrac{1}{Z_{C3}} + \dfrac{2}{Z_{C4}+R_4}}} + \frac{1}{Z_{C3} + \dfrac{1}{\dfrac{1}{R_3} + \dfrac{2}{Z_{C4}+R_4}}} \\
Y_C &= \frac{1}{R_3} + \frac{1}{Z_{C3}} + \frac{2}{Z_{C4}+R_4} \\
Y_D &= \frac{1}{R_4} + \frac{1}{Z_{C4}}
\end{aligned}
\tag{12}
$$

After simplifying Eq. (11), the output voltages are given by

$$
V_{out1,2,3,4} = \frac{V_{in1,2,3,4}\left(\begin{aligned}&Z_{C4}\left(\begin{aligned}&Z_{C3}\left\{Z_{C2}\left[Z_{C1}+R_1\left(\frac{V_{in4,1,2,3}}{V_{in1,2,3,4}}\right)\right]+\left(\frac{V_{in4,1,2,3}}{V_{in1,2,3,4}}\right)R_2\left[Z_{C1}+R_1\left(\frac{V_{in3,4,1,2}}{V_{in4,1,2,3}}\right)\right]\right\}\\&+\left(\frac{V_{in4,1,2,3}}{V_{in1,2,3,4}}\right)R_3\left\{Z_{C2}\left[Z_{C1}+R_1\left(\frac{V_{in3,4,1,2}}{V_{in4,1,2,3}}\right)\right]+\left(\frac{V_{in3,4,1,2}}{V_{in4,1,2,3}}\right)R_2\left[Z_{C1}+R_1\left(\frac{V_{in2,3,4,1}}{V_{in3,4,1,2}}\right)\right]\right\}\end{aligned}\right)\\&+\left(\frac{V_{in4,1,2,3}}{V_{in1,2,3,4}}\right)R_4\left(\begin{aligned}&Z_{C3}\left\{Z_{C2}\left[Z_{C1}+R_1\left(\frac{V_{in3,4,1,2}}{V_{in4,1,2,3}}\right)\right]+\left(\frac{V_{in3,4,1,2}}{V_{in4,1,2,3}}\right)R_2\left[Z_{C1}+R_1\left(\frac{V_{in2,3,4,1}}{V_{in3,4,1,2}}\right)\right]\right\}\\&+\left(\frac{V_{in3,4,1,2}}{V_{in4,1,2,3}}\right)R_3\left\{Z_{C2}\left[Z_{C1}+R_1\left(\frac{V_{in2,3,4,1}}{V_{in3,4,1,2}}\right)\right]+\left(\frac{V_{in2,3,4,1}}{V_{in3,4,1,2}}\right)R_2\left[Z_{C1}+R_1\left(\frac{V_{in1,2,3,4}}{V_{in2,3,4,1}}\right)\right]\right\}\end{aligned}\right)\end{aligned}\right)}{\left\{\begin{aligned}&\left[2R_1Z_{C1}+\left(Z_{C2}+R_2\right)\left(Z_{C1}+R_1\right)\right]\left[2R_3Z_{C3}+\left(Z_{C3}+R_3\right)\left(Z_{C4}+R_4\right)\right]\\&+2\left(Z_{C3}+R_3+Z_{C4}+R_4\right)\left[\left(Z_{C1}+R_1\right)R_2Z_{C2}+\left(Z_{C2}+R_2\right)R_1Z_{C1}\right]\end{aligned}\right\}}
\tag{13}
$$

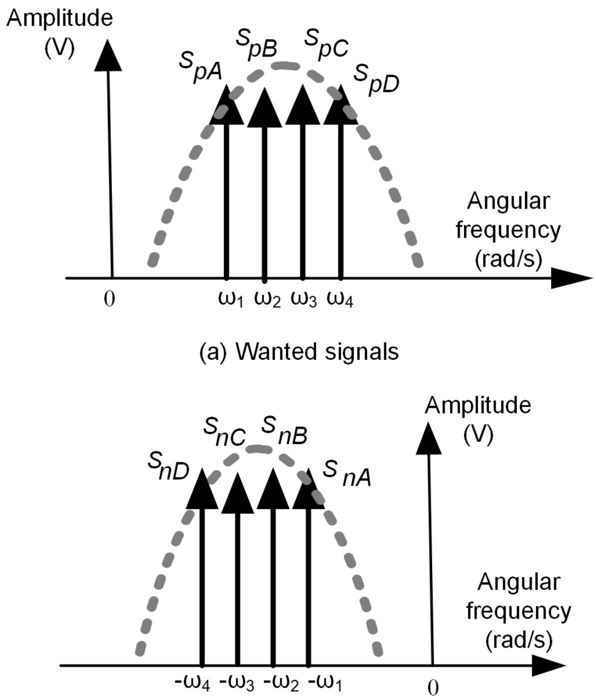
4.2. Mathematical model of transfer function for 4-stage RC Network

In this section, a mathematical model of the transfer function is proposed for the 4-stage RC polyphase filter. The properties of the transfer function of the RC polyphase network depend on the relationship of phases of the input sources.

Let $\omega_1 = \dfrac{1}{R_1C_1}, \omega_2 = \dfrac{1}{R_2C_2}, \omega_3 = \dfrac{1}{R_3C_3}, \omega_4 = \dfrac{1}{R_4C_4}$ be the selected cutoff angular frequencies of the 4-stage RC network. Assume that $S_{pA}, S_{pB}, S_{pC}, S_{pD}$ are the selected positive polyphase signals of the wanted signals as shown in Fig. 9(a). Their phase relationships are a negative notation of *j* (or *–j*). In case of image signals, $S_{nA}, S_{nB}, S_{nC}, S_{nD}$ are the selected negative polyphase signals as shown in Fig. 9(b). Their phase relationships are a positive notation of *j* (or *+j*). The image signals are expressed on the negative angular frequency plane and rejected by the polyphase filter.

In case of input positive polyphase signal, the transfer function is calculated as

$$H_p = \frac{V_{out1,2,3,4}}{V_{in1,2,3,4}} = \frac{(Z_{C4} - jR_4)(Z_{C3} - jR_3)(Z_{C2} - jR_2)(Z_{C1} - jR_1)}{\left\{\begin{array}{l}\left[2R_1Z_{C1} + (Z_{C2} + R_2)(Z_{C1} + R_1)\right]\left[2R_3Z_{C3} + (Z_{C3} + R_3)(Z_{C4} + R_4)\right] \\ +2(Z_{C3} + R_3 + Z_{C4} + R_4)\left[(Z_{C1} + R_1)R_2Z_{C2} + (Z_{C2} + R_2)R_1Z_{C1}\right]\end{array}\right\}} \tag{14}$$



(a) Wanted signals

(b) Image signals

Fig. 9. Selected cut-off angular frequencies on the angular frequency planes.

In case of the input negative polyphase signal, the transfer function is calculated as:

$$H_n = \frac{V_{out1,2,3,4}}{V_{in1,2,3,4}} = \frac{(Z_{C4} + jR_4)(Z_{C3} + jR_3)(Z_{C2} + jR_2)(Z_{C1} + jR_1)}{\left\{\begin{array}{l}\left[2R_1Z_{C1} + (Z_{C2} + R_2)(Z_{C1} + R_1)\right]\left[2R_3Z_{C3} + (Z_{C3} + R_3)(Z_{C4} + R_4)\right] \\ +2(Z_{C3} + R_3 + Z_{C4} + R_4)\left[(Z_{C1} + R_1)R_2Z_{C2} + (Z_{C2} + R_2)R_1Z_{C1}\right]\end{array}\right\}} \tag{15}$$

On the positive frequency domain, the transfer functions of the RC polyphase network in case of the wanted signal and the image signal are different. In this paper, to analyze the characteristics of the transfer function of the 4-stage RC network in both cases, the frequency responses of Eq. (14) and Eq. (15) are added in all frequency domains. We proposed a mathematical model of the transfer function of the polyphase filter which can be plotted on all frequency domains, from minus infinity to plus infinity as shown in Eq. (16). Compare the characteristics of the proposed model on all frequency domains with the characteristics of two transfer functions which are only plotted on the positive frequency domain, and we see that they are the same. The simplified transfer function is

$$|H(f)| = \frac{\left|f^4 + a_1f^3 + a_2f^2 + a_3f + a_4\right|}{\sqrt{\left(f^4 + a_4 - a_6f^2\right)^2 + \left(a_7f - a_5f^3\right)^2}}; \forall f \in R \tag{16}$$

Here, the frequencies on Eq. (16) are defined as

$$\begin{aligned}&f_1=\frac{1}{2\pi R_1C_1};f_2=\frac{1}{2\pi R_2C_2};f_{21}=\frac{1}{2\pi R_2C_1};f_3=\frac{1}{2\pi R_3C_3};f_{31}=\frac{1}{2\pi R_3C_1};f_{32}=\frac{1}{2\pi R_3C_2};\\&f_4=\frac{1}{2\pi R_4C_4};f_{41}=\frac{1}{2\pi R_4C_1};f_{42}=\frac{1}{2\pi R_4C_2};f_{43}=\frac{1}{2\pi R_4C_3};\end{aligned}\tag{17}$$

To simply the transfer function, the values of the given frequencies are expressed as:

$$\begin{aligned}a_1&=f_4+f_3+f_2+f_1\\a_2&=f_4\left(f_3+f_2+f_1\right)+f_3\left(f_2+f_1\right)+f_2f_1\\a_3&=f_4\left[f_3\left(f_2+f_1\right)+f_2f_1\right]+f_3f_2f_1\\a_4&=f_4f_3f_2f_1\\a_5&=a_1+2\left[\left(f_{43}+f_{42}+f_{41}\right)+\left(f_{32}+f_{31}\right)+f_{21}\right]\\a_6&=a_2+4f_{43}f_{21}+2\begin{pmatrix}f_4\left(\left(f_{32}+f_{31}\right)+f_{21}\right)+f_3\left(\left(f_{42}+f_{41}\right)+f_{21}\right)\\+f_2\left(\left(f_{43}+f_{41}\right)+f_{31}\right)+f_1\left(\left(f_{43}+f_{42}\right)+f_{32}\right)\end{pmatrix}\\a_7&=a_3+2\left(f_4\left(f_3f_{21}+f_2f_{31}+f_1f_{32}\right)+f_3\left(f_2f_{41}+f_1f_{42}\right)+f_2f_1f_{43}\right)\end{aligned}\tag{18}$$

To find the maximum values of this transfer function, applyCauchy-Schwarz inequality theorem

$$a,b>0,a^2+b^2\geq 2ab;\ \min\left(a^2+b^2\right)=2ab\ as\ "a=b"\tag{19}$$

Then, apply this definition into the above positive transfer function and we have the following:

$$\left(f^4+a_4-a_6f^2\right)^2+\left(a_7f-a_5f^3\right)^2\geq 2\left|\left(f^4+a_4-a_6f^2\right)\right|\left|\left(a_7f-a_5f^3\right)\right|\tag{20}$$

So, the maximum values of Eq. (20) are found as

$$\left|\left(f^4+a_4-a_6f^2\right)\right|=\left|\left(a_7f-a_5f^3\right)\right|;\Rightarrow\begin{bmatrix}f_{\max 1}=\ldots\ \Rightarrow\left|H\left(f\right)\right|_{\max 1}\\f_{\max 2}=\ldots\ \Rightarrow\left|H\left(f\right)\right|_{\max 2}\end{bmatrix}\tag{21}$$

On the positive frequency domain, the characteristics of the equation Eq. (16) are

$$
\begin{aligned}
&\left|H_p(f)\right| = \frac{f^4 + a_1 f^3 + a_2 f^2 + a_3 f + a_4}{\sqrt{\left(f^4 + a_4 - a_6 f^2\right)^2 + \left(a_7 f - a_5 f^3\right)^2}}; \forall f > 0 \\
&\lim_{f\to 0}\left|H_p(f)\right| = \lim_{f\to 0}\frac{f^4 + a_1 f^3 + a_2 f^2 + a_3 f + a_4}{\sqrt{\left(f^4 + a_4 - a_6 f^2\right)^2 + \left(a_7 f - a_5 f^3\right)^2}} = 1 \\
&\lim_{f\to \infty}\left|H_p(f)\right| = \lim_{f\to \infty}\frac{f^4 + a_1 f^3 + a_2 f^2 + a_3 f + a_4}{\sqrt{\left(f^4 + a_4 - a_6 f^2\right)^2 + \left(a_7 f - a_5 f^3\right)^2}} = 1 \\
&\min\left(\left|H_p(f)\right|\right) as\ f = \sqrt[4]{f_4 f_3 f_2 f_1} \\
&\max 1\left(\left|H_p(f)\right|\right) as\ f_{\max 1} = \ldots\ ; \qquad \max 2\left(\left|H_p(f)\right|\right) as\ f_{\max 2} = \ldots
\end{aligned}
\tag{22}
$$

On the positive frequency domain, there are one local minimum value at $f_{\min} = \sqrt[4]{f_1 f_2 f_3 f_4}$ and two local maximum values at $f_{\max 1}$ and $f_{\max 2}$. The gain ripple of the pass-band is measured from the local maximum value to the local minimum value.

In case of negative frequency, the properties of equation Eq. (16) are defined as

$$
\begin{aligned}
&\left|H_n(f)\right| = \frac{f^4 - a_1 f^3 + a_2 f^2 - a_3 f + a_4}{\sqrt{\left(f^4 + a_4 - a_6 f^2\right)^2 + \left(a_7 f - a_5 f^3\right)^2}}; \forall f < 0 \\
&\lim_{f\to 0}\left|H_n(f)\right| = \lim_{f\to 0}\frac{f^4 - a_1 f^3 + a_2 f^2 - a_3 f + a_4}{\sqrt{\left(f^4 + a_4 - a_6 f^2\right)^2 + \left(a_7 f - a_5 f^3\right)^2}} = 1 \\
&\lim_{f\to -\infty}\left|H_n(f)\right| = \lim_{f\to -\infty}\frac{f^4 - a_1 f^3 + a_2 f^2 - a_3 f + a_4}{\sqrt{\left(f^4 + a_4 - a_6 f^2\right)^2 + \left(a_7 f - a_5 f^3\right)^2}} = 1 \\
&\min\left(\left|H_n(f)\right|_{dB}\right) = -\infty; as\ f = -f_1 \vee f = -f_2 \vee f = -f_3 \vee f = -f_4 \\
&\max 1\left(\left|H_n(f)\right|\right) as\ f_{\max 1} = \sqrt{f_1 f_2}; \qquad \max 2\left(\left|H_n(f)\right|\right) as\ f_{\max 2} = \sqrt{f_2 f_3} \\
&\max 3\left(\left|H_n(f)\right|\right) as\ f_{\max 3} = \sqrt{f_3 f_4}
\end{aligned}
\tag{23}
$$

On the negative frequency domain, there are four local minimum values at cutoff frequencies $-f_1$, $-f_2$, $-f_3$, $-f_4$ and three local maximum values at $f_{\max 1} = \sqrt{f_1 f_2}$, $f_{\max 2} = \sqrt{f_2 f_3}$ and $f_{\max 3} = \sqrt{f_3 f_4}$. The gain ripple of the rejection-band is measured from the local maximum value to the local minimum value.

The cutoff frequencies for notch filters (at four local minimum values) are determined based the gain ripple of the rejection-band and the image rejection ratio.

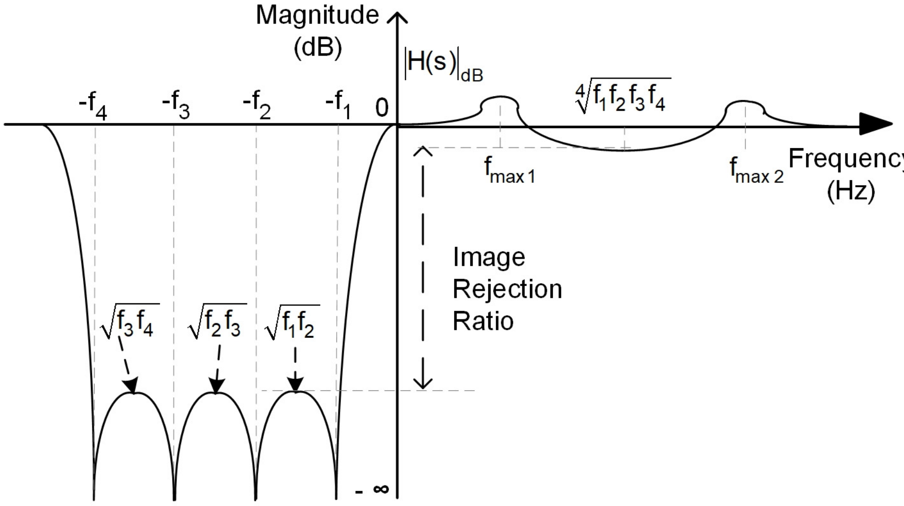


Fig. 10. Image rejection ratio of 4-stage RC polyphase filter.

4.3. Image rejection ratio of 4-Stage passive RC polyphase filter

On the frequency domain, the transfer function is plotted on Fig. 10. Image rejection ratio (IRR) is defined by the ratio of a notch band and a pass band of the network [12]. Using Eq. (14) and Eq. (15), the image signal rejection ratio ofthe 4-stage RC network is defined as:

$$IRR = \frac{H_{Pos}(j2\pi f)}{H_{Neg}(j2\pi f)} = \frac{(f_1+f)(f_2+f)(f_3+f)(f_4+f)}{(f_1-f)(f_2-f)(f_3-f)(f_4-f)} \tag{24}$$

This ratio depends on the selected notch frequencies. However, the conversion gain on the positive frequency is not flat. In other words, it is not an ideal band-pass filter for the positive polyphase signals. So, the amplitude of the input signals will be changed according to the values of the selected frequencies.

5. Proposed design of 4-stage passive RC polyphase filter

5.1Mathematical model of 4-stage RC polyphase filter

On this research, the 4-stage RC network is modeled by the transfer function as shown in Eq. (25). To design a low area chip, the values of each component are chosen as: $R_1 = R_2 = R_3 = R_4 = 1$ kΩ, $C_1 = 227$ pF, $C_2 = 106$ pF, $C_3 = 39.8$ pF, $C_4 = 19.9$ pF. On the negative frequency domain, there are four local minimum values at cutoff frequencies $-f_1 = -0.7$ MHz, $-f_2 = -1.5$ MHz, $-f_3 = -4$ MHz, $-f_4 = -8$ MHz, and three local maximum values at $f_{max1} = -1.02$ MHz, $f_{max2} = -2.45$ MHz, and $f_{max3} = -6.35$ MHz. On the positive frequency domain, there are one local minimum value at $f_{min} = 2.5$ MHz and two local maximum values at $f_{max1} = 0.159$ MHz and $f_{max2} = 40$ MHz. The given variables $a_{1,2,3,4,5,6,7}$ in Eq. (23) are calculated as $a_1 = 1.42*10^7$, $a_2 = 5.95*10^{13}$, $a_3 = 8.3*10^{19}$, $a_4 = 3.36*10^{25}$, $a_5 = 3.24*10^7$, $a_6 = 1.66*10^{14}$, and $a_7 = 1.87*10^{20}$.

$$|H(f)|_{dB} = \frac{\left|f^4 + 1.42*10^7 f^3 + 5.95*10^{13} f^2 + 8.3*10^{19} f + 3.36*10^{25}\right|}{\sqrt{\left(f^4 + 3.36*10^{25} - 1.66*10^{14} f^2\right)^2 + \left(1.87*10^{20} f - 3.24*10^7 f^3\right)^2}} \tag{25}$$

On the positive and negative frequency domain, the characteristics of this filter are calculated in the Eq. (40).

$$\left|H_p(f)\right|_{dB} = \begin{cases} 0dB;\ f \to 0 \\ -\infty;\ f = -7*10^5 \\ -35.6dB;\ f = -10.2*10^5 \\ -\infty;\ f = -15*10^5 \\ -34.8dB;\ f = -24.5*10^5 \\ -\infty;\ f = -40*10^5 \\ -32.1dB;\ f = -63.2*10^5 \\ -\infty;\ f = -80*10^5 \\ 0dB;\ f \to -\infty \end{cases} \qquad \left|H_n(f)\right|_{dB} = \begin{cases} 0dB;\ f \to 0 \\ 1.33dB;\ f = 1.59*10^5 \\ -0.61dB;\ f = 2.5*10^6 \\ 1.38dB;\ f = 40*10^6 \\ 0dB;\ f \to \infty \end{cases} \tag{26}$$

5.2. Matlab simulation of 4-stage RC polyphase filter

This section is focused on Matlab simulation of the proposed model of the 4-stage RC polyphase filter. In SPICE simulation, it is difficult to plot the negative frequency response of the network. The magnitude plot of the 4-stage RC polyphase filter is shown in Fig. 11. The image signal rejection ratio of this model is 36 dB. It is properly considered for a narrow band low-IF receiver.

The Matlab code of this model is written as below

```
% Frequency range from -10 MHz to 10 MHz
f= [-10000000:1000:10000000];
% First cutoff frequency at 0.7 MHz
R1=1e3; C1=227e-12; f1=1/(2*pi*R1*C1);
% Second cutoff frequency at 1.5 MHz
R2=1e3; C2=106e-12; f2=1/(2*pi*R2*C2);
f21=1/(2*pi*R2*C1);
% Third cutoff frequency at 4 MHz
R3=1e3; C3=39.8e-12; f3=1/(2*pi*R3*C3);
f31=1/(2*pi*R3*C1);f32=1/(2*pi*R3*C2);
% Fourth cutoff frequency at 8 MHz
R4=1e3; C4=19.9e-12; f4=1/(2*pi*R4*C4);
f41=1/(2*pi*R4*C1);f42=1/(2*pi*R4*C2); f43=1/(2*pi*R4*C3);
% Definitions of the constant variables a1, a2, a3, a4, a5, a6, a7
a1=f1+f2+f3+f4;
a2=f4*(f3+f2+f1)+f3*(f2+f1)+f2*f1;
a3=f4*(f3*(f2+f1)+ f2*f1)+ f3*f2*f1;
a4=f1*f2*f3*f4;
a5=a1+2*(f43+f42+f41+f32+f31+f21)
a6=a2+2*(2*f43*f21+f4*((f32+f31)+f21)+f3*((f42+f41)+f21)
    +f2*((f43+f41)+f31)+f1*((f43+f42)+f32));
a7=a3+2*(f4*(f3*f21+f2*f31+f1*f32)+f3*(f2*f41+f1*f42)+f2*f1*f43);
% Definition of the transfer function
```

```
Num=(f.^4 +f.^3*a1+ f.^2*a2+ f*a3+ a4); % the numerator
Den=(sqrt((f.^4+a4-f.^2*a6).^2 +(f*a7-f.^3*a5).^2)); % the denominator
H= Num./ Den; % the transfer function
% Definition of the magnitude plot of the transfer function
AMP_H=20*log10(abs(H)); %decibel value of the magnitude
% Plot the transfer function
close all;
plot(f,AMP_H); grid on;
xlabel('Frequency [Hz]'); ylabel('Magnitude [dB]');
```

5.3. SPICE simulation of proposed design

In this section, the proposed design of the 4-stage RC polyphase filter is presented on SPICE simulation. Fig. 12(a) shows the SPICE circuit of the proposed design. On this model, the parasitic of RC components and the time delay of the signal propagation are neglected. Both positive and negative polyphase signals are used to do the simulation. The image signal rejection ratio of this filter is 36 dB as shown in Fig. 12(b).

Compare to the SPICE simulation result with the Matlab simulation result, and we see that they are the same. There are two local maximum values on positive frequency and four minimum values on negative frequency. The ripple gain is 2 dB on the positive frequency domain at 160 kHz (1.24 dB) and 40 MHz (1.24 dB).

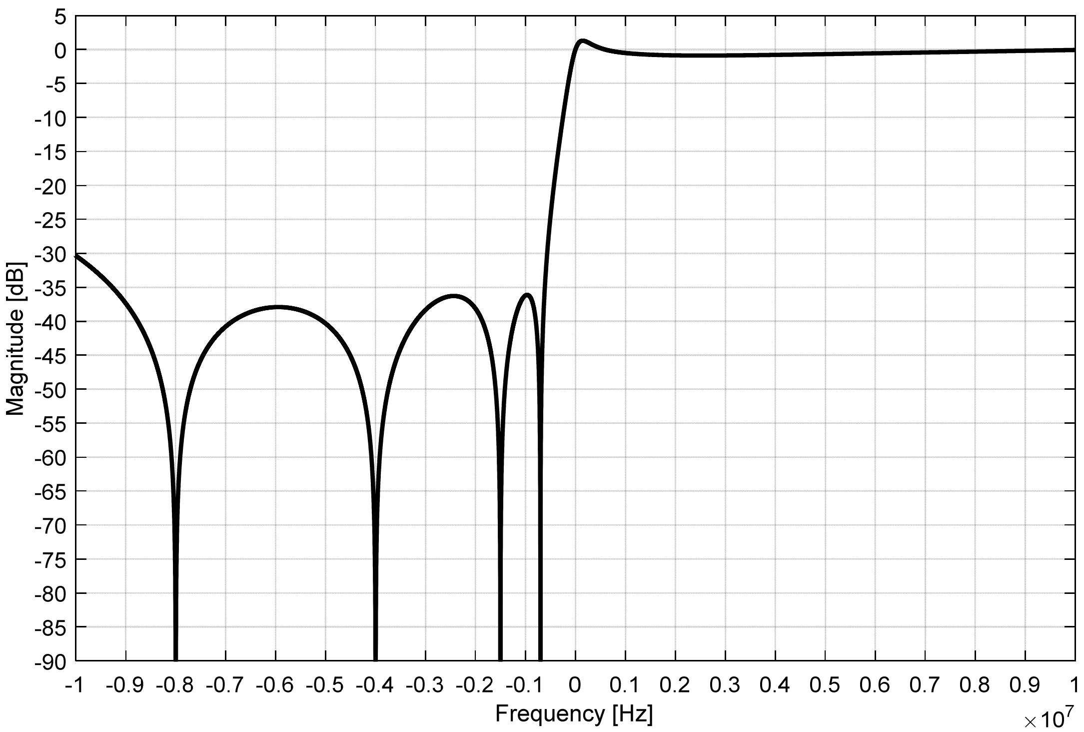
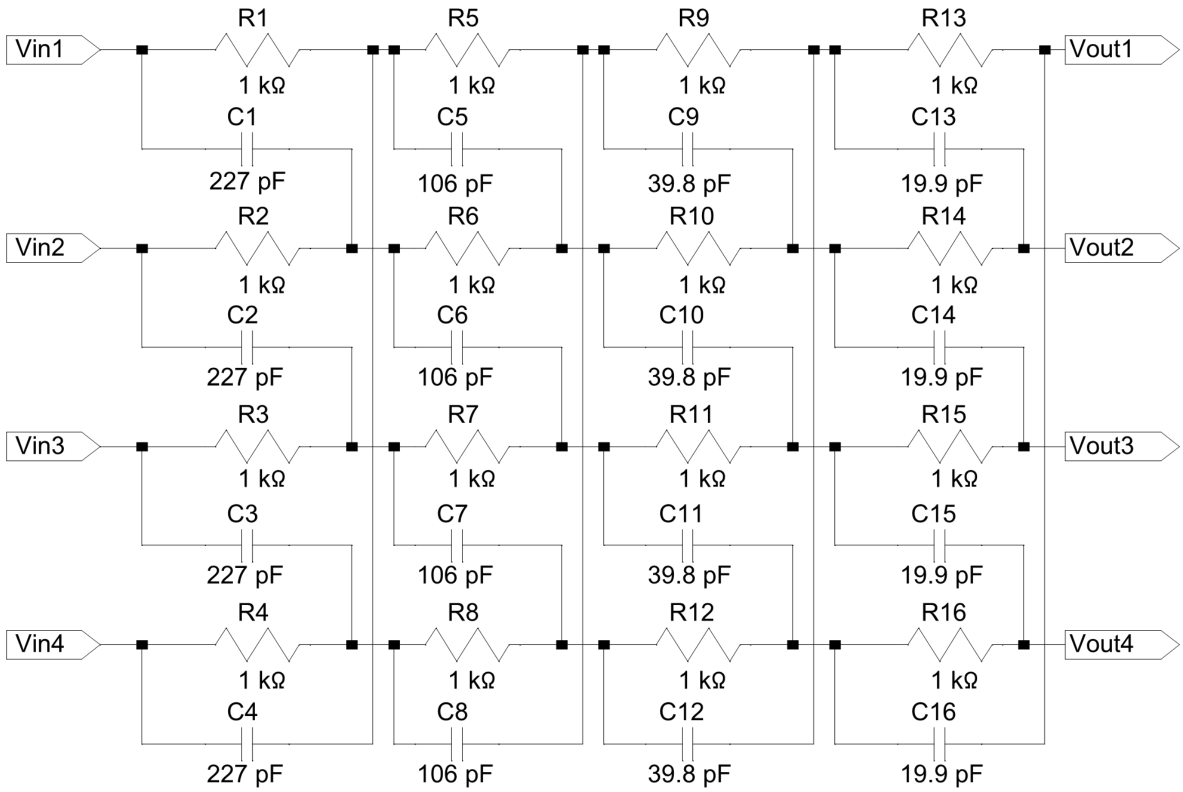
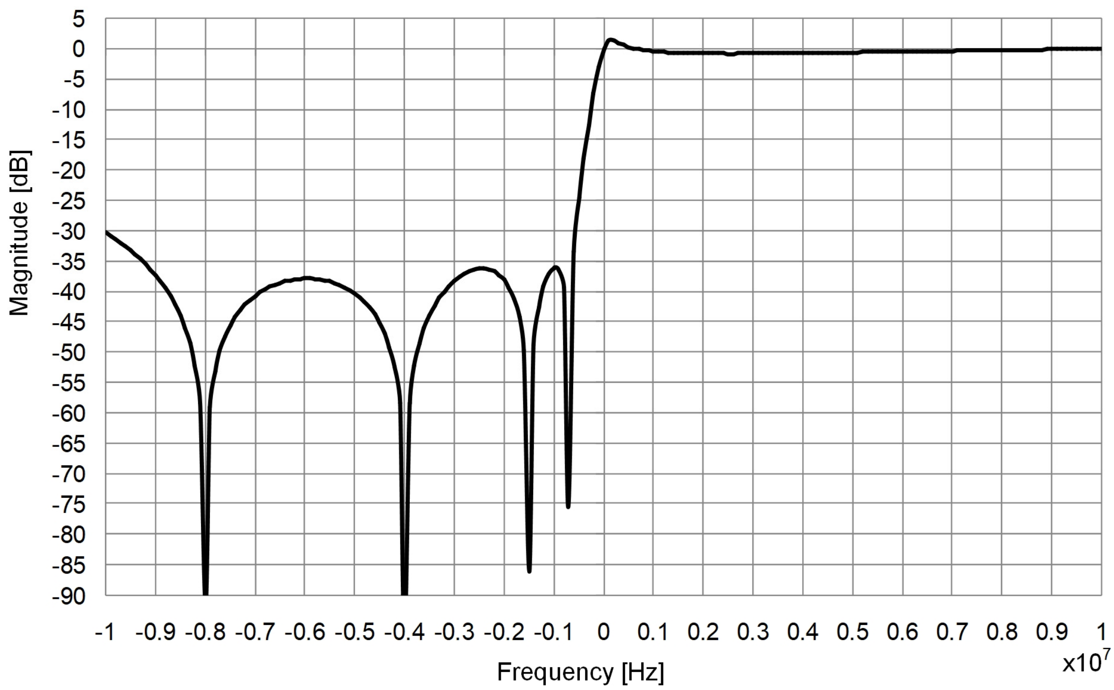


Fig. 11. Matlab simulation of 4-stage RC polyphase filter in all frequency domains.



(a) SPICE model of 4-stage RC polyphase circuit



(b) SPICE simulation result of 4-stage RC polyphase filter

Fig. 12. SPICE simulation result of proposed design of 4-stage RC polyphase filter.

6. Discussion

There are many trade-off factors when a polyphase filter is designed such as the input/output matching impedance, the noise figure, the power loss, the cost and semiconductor technology, layout technique and so on... To keep small gain ripple of rejection-band and high image rejection ratio (signal to noise = image rejection ratio = 1000), the cut-off frequencies of each stage are separated at different notch frequencies from low to high frequencies.

Due to the application and the required image rejection ratio, the number of stages in a polyphase filter is different [13]. A 3-stage RC network can be sufficient to realize 30 dB rejection ratio with 10 MHz rejection-band on the negative frequency domain. However, the gain ripples of the pass-band and the rejection-band in the 3-stage RC network are larger than the gain ripples in 4-stage RC network. To realize 30 dB (10*log(1000) = 30 dB) image rejection ratio for 10 MHz rejection-band (from -0.5 MHz to -10.5 MHz), 4-stage passive RC network is proposed to design the polyphase filter.

7. Conclusions

In theory design, the low pass filters and the complementary high pass filters are used to constitute a quadratic filter. Positive and negative Hilbert transforms are used to clarify the direction of two sets of phasors which are used to express the wanted signals and the image signals on the phase-order planes. The concepts of positive and negative polyphase signals are also used to define the phase directions of these signals.

The angular frequency variables ω and -ω are proposed to represent these received signals on the angular frequency plane. The widened superposition principle is used to derive the transfer function of a 4-stage passive polyphase circuit. A model of the transfer function of the 4-stage RC network is mathematically analyzed and plotted on all frequency domains. Moreover, the values of resistors and capacitors are chosen based on small chip area. As a result, the image rejection ratio of this network is 36 dB which is sufficient for 10 MHz bandwidth system. The obtained results were acquired by simulations using SPICE models of the devices. Compared to the research results in Matlab simulation with the SPICE simulation of this filter, they are the same. In this paper not only the results of the mathematical model but also the results of simulation of the designed circuits are provided. The simulation results and the values of theoretical calculation of the transfer function are unique. In future work, the parasitics of RC components and the amplitude/phase mismatches in these systems will be analyzed.

Acknowledgement

The authors would like to thank Prof. Hiroshi Tanimoto for valuable comments.

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