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Analysis and Experiments on Nonlinear Vibrations of a Post-buckled Beam with a Stepped Section

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Abstract. Analytical and experimental results are presented on nonlinear vibrations of a post-buckled beam with a stepped section constrained by an axial elastic spring. In the experiment, the beam is excited laterally under periodic acceleration, and the dynamic responses of the beam are measured. In the analysis, the beam is divided into a few segments. The deflection of the beam is expanded with the mode shape function that is expressed with the product of truncated power series and trigonometric functions. Taking the axial displacement, the deflections, slopes, bending moments and shearing forces at the nodes of the segments as unknown variables, nonlinear coupled ordinary differential equations are derived with the Galerkin procedure. Neglecting the axial inertia of the beam, the axial displacements at the nodes are expressed as nonlinear functions of the deflections, slopes, bending moments and shearing forces, nonlinear responses are calculated with the harmonic balance method and with the direct time integration. Fairly good agreements are obtained between results of experiment and analysis.

1. Introduction

Recently, technology of a micro electro-mechanical system (MEMS) has been developed drastically. Micro devices such as an acceleration pickup and an optical scanner are widely utilized. These devices are composed with elements of thin elastic structures. The elements have complicated shape with discontinuous cross section like a stepped beam or combined configuration of beam and plate. When the thin beams are subjected to periodic force and large amplitude resonance are generated, nonlinear responses are easily generated [1]. Therefore, in this paper, both experimental and analytical results are presented on nonlinear vibrations of a stepped beam. The rectangular cross section of the beam is changed to H-shape at the mid span of the beam. One end of the beam is clamped and the other is simply supported. The beam is compressed to the post-buckled state by the spring in the axial direction. In the experiment, the beam is excited laterally under periodic acceleration, and the dynamic responses of the beam are measured. In the analysis, the beam is divided into a few segments. The deflection of the beam is expanded with the mode shape function that is expressed with the product of truncated power series and trigonometric functions [2]. Taking the axial displacement, the deflections, slopes, bending moments and shearing forces at the nodes of the segments as unknown variables, nonlinear coupled ordinary differential equations are derived with the Galerkin procedure. Neglecting the axial inertia of the beam, the axial displacements at the nodes are expressed as nonlinear functions of the deflections, slopes, bending moments and shearing forces, nonlinear responses are calculated with the harmonic balance method and with the direct time integration.

2. Procedure of Experiment

Fig. 1 shows the stepped beam and its fixture. A thin phosphor bronze beam with thickness h=0.30 mm, breadth b=40 mm and length L=140 mm is clamped at one end and simply-supported at the other end. Four thin phosphor bronze beam (thickness 0.31 mm, breadth 4.9 mm, length 34 mm) are attached to the mid span of the beam, then the cross section is locally changed to H-shaped. At the simply-supported end, the beam is connected to an elastic plate by the strips of adhesive films. The elastic plate is clamped by the slide block and works as the axial spring. The beam is compressed by the axial spring, then the beam is deformed to the post-buckled configuration. To find fundamental properties of the beam, the linear natural frequencies and the restoring force are inspected. The post-buckled beam is excited laterally with an electromagnetic exciter. The beam is subjected to gravitational acceleration and periodic acceleration $a_d \cos 2\pi ft$, where f is the excitation frequency and a_d is the peak amplitude of acceleration.

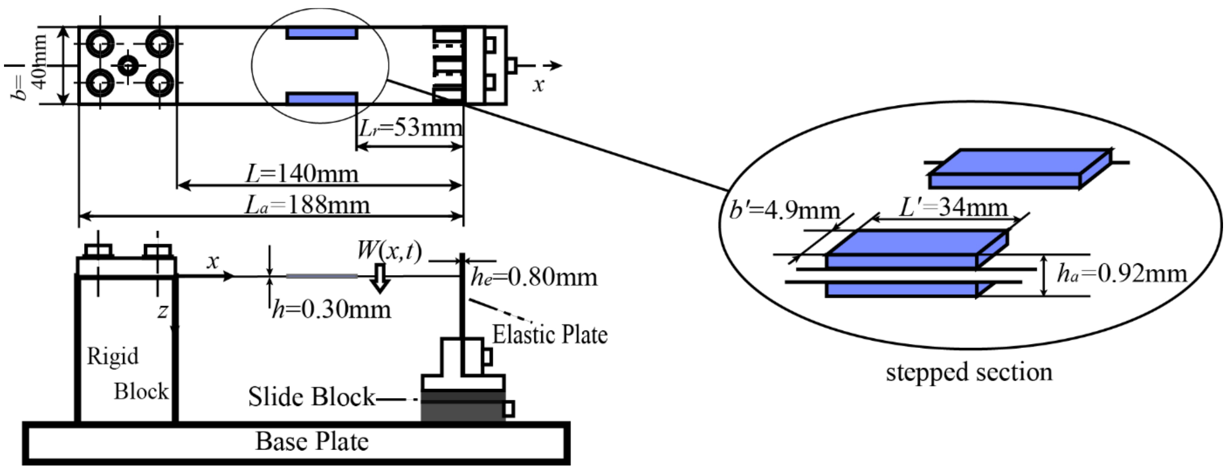

Fig. 1. Beam and fixture

3. Procedure of Analysis

Fig. 2 shows the analytical model of the post-buckled beam elastically constrained at an end. We introduce the x and z axes along the axial and lateral directions of the beam, respectively. The origin is taken at the fixed end of the beam. The symbols L and K denote the length of the beam and the spring constant of the axial spring, respectively. The beam is divided into N=3 segments, two of which correspond to the parts with original rectangular cross section, the other corresponds to the part with H-shaped cross section.

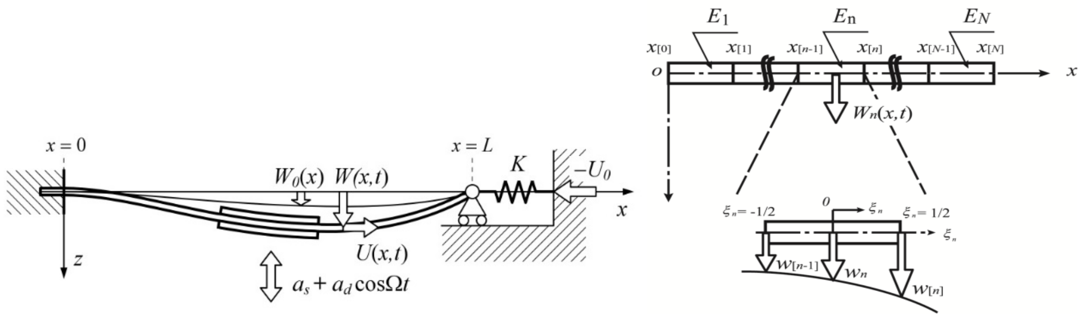

Fig. 2. Analytical model of a stepped beam

The local coordinate in the n-th segment x_n is introduced which spans from x_n =-1/2 to x_n =1/2. The length, mass density, Young's modulus, area and moment of a cross section of the n-th segment are expressed by l_n, ρ_n, E_n, A_n and I_n, respectively. The beam is subjected to the static and periodic acceleration $a_s+a_d\cos\Omega t$. The deflection and axial displacement of the beam is expressed as $W(x,t)$ and $U(x,t)$, respectively. The beam is buckled with the initial axial displacement of the axial spring U_0. For sufficiently thin beams, the axial inertia of beam itself, rotational inertia and shearing deformation can be neglected. The non-dimensional governing equation of the vibrations of the beam is expressed as follow, with the Hamilton's principle.

$$\int_{\tau_o}^{\tau_1}\left\{\sum_{n=1}^{N}\left[\int_{-1/2}^{1/2}G_w\left(w_n\right)\delta w_n d\xi_n+\left[q_{xn}\delta w_n\right]_{-1/2}^{1/2}-\left[d_n m_{xn}\delta w_{n,\xi_n}\right]_{-1/2}^{1/2}+\left[n_{xn}\delta u_n\right]_{-1/2}^{1/2}\right]\right.$$
$$\left.+k\left(u_{[N]}-u_0\right)\delta u_{[N]}\right\}d\tau=0 \tag{1a}$$

$$G_w\left(w_n,u_n\right)=d_n^{-1}\overline{\rho_n}\,\overline{A_n}w_{n,\tau\tau}-d_n n_{xn}w_{n,\xi_n\xi_n}-d_n m_{xn,\xi_n\xi_n}-d_n^{-1}\bar{\rho}_n\bar{A}_n p-\frac{1}{d_n}\delta\left(\xi-\xi_s\right)q_s \tag{1b}$$

$$n_{xn}=d_n\bar{E}_n\bar{A}_n\left(u_{[n]}-u_{[n-1]}\right)+\frac{1}{2}d_n^2\bar{E}_n\bar{A}_n\int_{-1/2}^{1/2}\left(w_{n,\xi_n}^{2}-w_{0n,\xi_n}^{2}\right)d\xi_n \tag{1c}$$

In the above equation, d_n is defined as $d_n=L/l_n$, the symbols w_n, u_n, u_0, n_{xn}, s_{xn}, m_{xn} and q_{xn} are non-dimensional deflection, axial displacement, initial axial displacement, axial force, slope, bending moment and shearing force, respectively. Non-dimensional lateral acceleration is denoted by $p=p_s+p_d\cos\omega\tau$, ω and τ are non-dimensional excitation frequency and time, k is the non-dimensional axial spring constant. A vector $\{w_{en}\}$ that consists of nodal deflection w_n, slope s_{xn}, bending moment m_{xn} and shearing force q_{xn} at the both nodes of the n-th segment is introduced, then the deflection w_n in the n-th segment is expressed with the coordinate function $\{\zeta_n\}$, following the similar manner of the finite element procedure.

$$w_n\left(\xi_n,\tau\right)=\left\{\zeta_n\right\}^T\left\{w_{en}\right\}=\sum_{j=1}^{8}w_{enj}\left(\tau\right)\zeta_{nj}\left(\xi_n\right),\quad \left\{\zeta_n\right\}=\left\{\bar{Z}_n\right\}^T\left(\left[D_n\right]\left[Z_n\right]\right)^{-1},$$
$$\bar{Z}_{ni}=\sum_{l=1}^{2}\sum_{k=1}^{4}\delta_{i,f(k,l)}\left\{\left(2\xi_n\right)^{k-1}\cos\left(l-1\right)\pi\left(\xi_n+1/2\right)\right\},f\left(k,l\right)=4(l-1)+k \tag{2}$$

In the above equations, $\{\bar{Z}_n\}$ is a vector composed of the mode shape function $\bar{Z}_{ni}$ that is the product of truncated power series and trigonometric functions, $[Z_n]$ is a 8×8 matrix consists of $\bar{Z}_{ni}$ and its first, second and third order derivatives, $[D_n]$ is a 8×8 matrix consists of parameters of the n-th segment. Introducing the global nodal vector $\{b\}$ which includes the nodal vector $\{w_{en}\}$ of the all segments, and the vector $\{d\}$ which consists of axial displacement of all nodes, and applying the Galerkin procedure, the nonlinear governing equation of the beam (Eq. (1)) is reduced to a set of ordinary differential equations as follows.

$$\sum_q \hat{B}_{pq}\hat{b}_{q,\tau\tau} + \sum_q \hat{C}_{pq}\hat{b}_q + \sum_q\sum_v \hat{D}_{pqv}\hat{b}_q\hat{d}_v + \sum_v\sum_q \hat{D}_{pvq}\hat{d}_v\hat{b}_q + \sum_q\sum_r\sum_s \hat{E}_{pqrs}\hat{b}_q\hat{b}_r\hat{b}_s - \hat{F}_p - \hat{G}_p\left(p_s + p_d\cos\omega\tau\right) = 0 \tag{3}$$

$$\sum_v \hat{C}_{tv}\hat{d}_v + \sum_r\sum_s \hat{D}_{trs}\hat{b}_r\hat{b}_s - \hat{F}_t = 0 \tag{4}$$

$$p,q,r,s = 1,2,\cdots,4(N+1), \quad t,v = 4(N+1)+1, 4(N+1)+2,\cdots,5(N+1)$$

Solving $\{\hat{d}\}$ in terms of $\{\hat{b}\}$ in Eq. (4), and then substituting it to Eq. (3), the axial displacements can be removed in the reduced governing equation. Neglecting the time variant terms, static deflection due to the static lateral acceleration and the axial initial displacements is obtained. Next, the ordinary differential equation is transformed to the equation in terms of the dynamic variable $\tilde{b}_j$ which is measured from the static equilibrium position. Furthermore, the ordinary differential equations are transformed to the standard form in terms of normal coordinates b_i corresponding to the linear natural modes of vibration $\tilde{\xi}_j$ at the static equilibrium position of the beam. Dynamic responses can be calculated with the harmonic balance method and the numerical integration.

4. Results and Discussion

Equivalent moment of cross section of the H-shaped part and the initial deflections are identified by comparing the experimental and analytical results of the post-buckled deformation (Fig. 3) and characteristics of restoring force under a concentrated lateral force on the beam (Fig. 4), for three conditions of the magnitude of axial compressive force. Fig. 5 shows the nonlinear frequency curves of the beam comparing the analytical and experimental nonlinear responses. In the figure, the black and gray curves are the stable and unstable periodic responses, respectively, calculated by the harmonic balance method. The principal resonance (1:1) and the sub-harmonic resonance (1:1/2) of the order 1/2 of the lowest mode appear corresponding to the softening-and-hardening characteristics of the restoring force. The results of direct numerical integration, shown with the blue curves, almost follow the stable periodic responses. Decreasing the no-dimensional exciting frequency ω from ω=35.00, the sub-harmonic resonance response (1:1/2) is bifurcated from the non-resonant response, at ω=28.00. The amplitude of the sub-harmonic resonance transits to the non-resonant response by a jump phenomenon at ω=20.67. As the frequency is decreased, the non-resonant response transits to the large amplitude of the principal resonance by a jump phenomenon at ω=11.35.

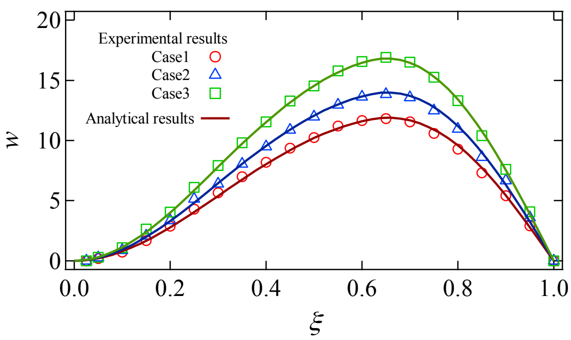

Fig. 3. Post-buckled deformation

Increasing the frequency from ω=11.35 at the large amplitude of the principal resonance, the large amplitude of the principal resonance transits to the non-resonant response by a jump phenomenon at ω=16.90. The red curves in the figure presents the experimental results. Fairly good agreements are obtained between experimental and analytical periodic responses.

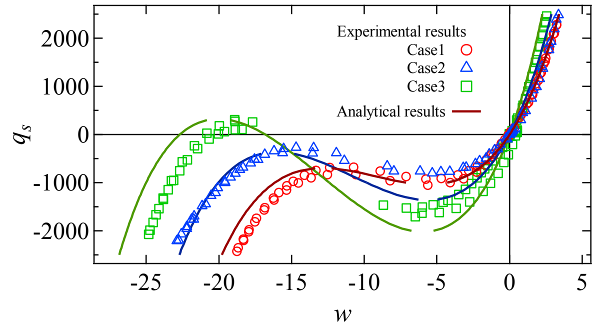

Fig. 4. Characteristics of restoring force

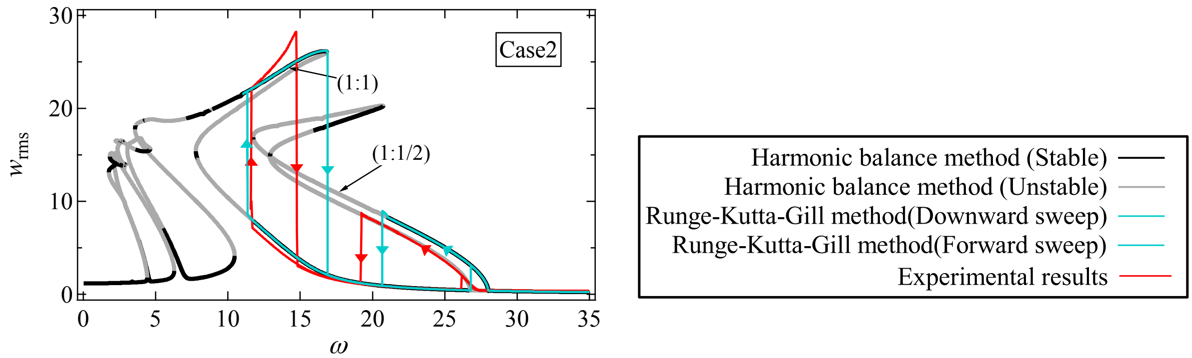

Fig. 5. Nonlinear frequency response curves

5. Conclusion

Analytical and experimental results are presented on nonlinear vibrations of a post-buckled beam with a stepped section constrained by an axial elastic spring. In the experiment, the beam is excited laterally under periodic acceleration, and the dynamic responses of the beam are measured. In the analysis, the beam is divided into a few segments. The deflection of the beam is expanded with the mode shape function that is expressed with the product of truncated power series and trigonometric functions. Taking the axial displacement, the deflections, slopes, bending moments and shearing forces at the nodes of the segments as unknown variables, nonlinear coupled ordinary differential equations are derived with the Galerkin procedure. Neglecting the axial inertia of the beam, the axial displacements at the nodes are expressed as nonlinear functions of the deflections, slopes, bending moments and shearing forces, which decreases the computational cost considering the coupling between deflection and axial deformation. Fairly good agreements are obtained between results of experiment and analysis, which verifies the present analysis.

References

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