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Nonlinear stiffness identification based on AR time series analysis using transient free oscillation

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Abstract. The authors proposed an identification method of a nonlinear spring coefficient using Auto-Regressive time series analysis and a method of Krilov-Bogoliubov-Metropolsky (KBM method) in a previous paper. Especially, the conventional works had independently deal with a case of second order nonlinearity and a case of third order nonlinearity. However, there is a problem that the conventional work does not consider the case of simultaneous occur the second and the third order nonlinearity. In this paper, the system identification method in an asymmetric nonlinear system included second-order and third-order nonlinearity is investigated. At first, the formulation of the identification problem is conducted. The identification problem is described using the KBM method. The regression problem is formulated using the difference between instantaneous natural angular frequency and linear natural angular frequency. Furthermore, the identification experiment is conducted using the numerical investigation based on the 4th Runge-Kutta method. As a result, the estimation values of our proposed method are good agreement with the true values. Moreover, the experimental identification is conduced using the cantilever system which is simultaneous subjected the magnetic force and the geometric nonlinear spring force. The estimated relationship between restoring force and displacement is good agreement with the statically measured results.

1. Introduction

Understanding and quantification of nonlinear stiffness characteristics is important for precisely evaluating infrastructure. Especially, in evaluation of complex bridge structure and ultrathin cylindrical shell, which consist of various nonlinearities [1, 2]. For efficient operation maintenance, non-destructive inspection technology is required. In particular, the quantification method of nonlinear stiffness characteristics has contributed to the establishment of high precise non-destructive inspection. In this research, identification technology of nonlinear stiffness characteristics is developed based on a transient free vibration.

Recently, nonlinear identification method based on the resonance decay method has been proposed [3]. Furthermore, the identification technology has been expanded by Bayesian inferences based on Markov Chain Monte Carlo method (MCMC) [4]. This method includes estimating instantaneous amplitude and instantaneous frequencies using a short time Fourier transform, and the nonlinear coefficient is estimated by regression formula of MCMC based on frequency amplitude relationships using second-order normal form theory.

In other approaches, the regression formula is based on multiple scales method [5], whereby the instantaneous amplitude, the instantaneous frequency and vibration center are estimated from raw time series data. In addition, the identification experiment is conducted in a thin cantilever, including concentrated mass. The thin cantilever is subjected to an axial force by gravity of concentrated mass. Therefore, the nonlinear stiffness characteristics of second order and third order appear in the deflection of cantilever. The nonlinear coefficients of second order and third order are obtained using the above identification method.

In previous work, identification method is based on AR time series analysis using the transient free vibration data [6-10]. Our proposed identification method consists of the regression formula using Krilov-Bogoliubov-Metropolsky (KBM) method, and a high precisely estimation of instantaneous frequency using Kalman filter and estimation of instantaneous amplitude using Hilbert transform. Typically, the conventional works treated the cases of second and third orders nonlinearity independently. However, the case of second and third order nonlinearity, was not considered. In this paper, the system identification method in asymmetric nonlinear system that includes second-order and third-order nonlinearity is investigated.

2. Formulation of identification problem

2.1 Analytical consideration based on KBM method

An oscillator model of a spring-mass-damper system subjected to nonlinear restoring force is represented in Fig. 1. The equation of motion is denoted by Eq. (1).

$$m\ddot{x} + c\dot{x} + kx + N(x) = 0 \tag{1}$$

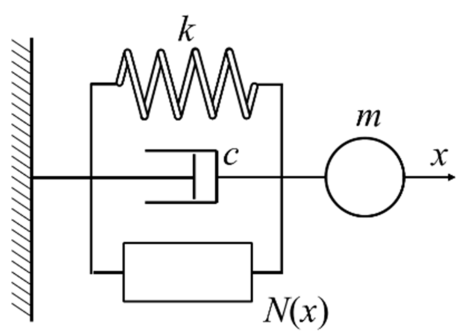


Fig. 1. Single degree-of-freedom system that contains the second and the third orders nonlinear spring.

Here, m, c, k represent the mass, damping constant, and spring constant, respectively. Moreover, $\ddot{x}, \dot{x}, x$ represent the acceleration, velocity and displacement of oscillator. The nonlinear term $N(x)$ considers the second-order term, λ, and the third-order term, μ. Therefore, $N(x) = \lambda x^2 + \mu x^3$. The normalized equation of motion by mass follows Eq. (2).

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x + e_2 x^2 + e_3 x^3 = 0, \qquad e_2 = \frac{\lambda}{m},\ \ e_3 = \frac{\mu}{m} \tag{2}$$

The amplitude and phase equations are obtained using Eqs. (3) and (4) by KBM method. Here, a represents the amplitude, and ψ represents the phase.

$$\frac{da}{dt} = -\zeta\omega_0 a + \frac{3\zeta e_3}{8\omega_0}a^3 \tag{3}$$

$$\frac{d\psi}{dt} = \omega_0\left(1 - \frac{\zeta^2}{2}\right) + \frac{3e_3}{8\omega_0}a^2 - \frac{1}{\omega_0^3}\left(\frac{5e_2^2}{12}a^2 + \frac{15e_3^2}{256}a^4\right) \tag{4}$$

The free vibration solution of second-order approximation is obtained by Eq. (5).

$$x = a\cos\psi - \frac{e_2 a^2}{2\omega_0^2} + \frac{e_2 a^2}{6\omega_0^2}\cos 2\psi + \frac{e_3 a^3}{32\omega_0^2}\cos 3\psi \tag{5}$$

2.2 Derivation of nonlinear stiffness regression formula

In this section, the nonlinear stiffness regression formula is derived using Eqs. (3) and (4). Here, the damping constant is neglected. The frequency-amplitude relationship is obtained by Eq. (6).

$$\omega^2(a) = \omega_0^2 + \frac{3e_3 a^2}{4} - \frac{5e_2^2 a^2}{6\omega_0^2} + \frac{3e_3^2 a^4}{128\omega_0^2} \tag{6}$$

During the transient free oscillation, we assume that there is a constant establishment of the frequency-amplitude relationship. Therefore, Eq. (7) can be obtained by the instantaneous amplitude $A(t)$ and the instantaneous frequency $\Omega(t)$.

$$\Omega(t) = \omega_0 + \frac{3e_3 A(t)^2}{8\omega_0} - \frac{1}{\omega_0^3}\left(\frac{5e_2^2 A(t)^2}{12} + \frac{15e_3^2 A(t)^4}{256}\right) \tag{7}$$

Here, a parameter, $\Delta\Omega(t)$, which is the difference between the instantaneous frequency $\Omega(t)$ and the linear eigen frequency ω_0 is introduced. Thus, the linear relationship of unknown parameter is derived.

$$\Delta\Omega(t) = \left(\frac{3e_3}{8\omega_0} - \frac{5e_2^2}{12\omega_0^3}\right)A(t)^2 - \frac{15e_3^2}{256\omega_0^3}A(t)^4 = \alpha A(t)^2 + \beta A(t)^4 \tag{8}$$

Furthermore, Least-Mean Square problem is defined using data matrix $\mathbf{X}, \mathbf{Y}$ based on the instantaneous amplitude and the instantaneous frequency.

$$\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y},\ \ \mathbf{Y} = \mathbf{C}(t),\ \ \mathbf{X} = [\mathbf{A}(t), \mathbf{B}(t)] \tag{9}$$
$$\mathbf{A}(t) = [A(t_0)^2, A(t_1)^2, \ldots, A(t_N)^2\ ]^T,\ \ \mathbf{B}(t) = [A(t_0)^4, A(t_1)^4, \ldots, A(t_N)^4]^T$$
$$\mathbf{C}(t) = [\Delta\Omega(t_0)^2, \Delta\Omega(t_1)^2, \ldots, \Delta\Omega(t_N)^2]^T$$

Therefore, the nonlinear stiffness parameter is obtained by Eqs. (10) and (11).

$$\lambda = \sqrt{\frac{12}{5}\omega_0^2\left(\frac{3}{8}\frac{\mu}{\omega_0 m} - \alpha\right)m^2} \tag{10}$$

$$\mu = \sqrt{-\frac{256}{15}\omega_0^3 \beta m^2} \tag{11}$$

In asymmetrical nonlinear system, the vibration center motion occurs. Therefore, the sign of average value of time series changes. The sign discrimination function of nonlinear coefficient λ using the average value of time series is expressed by Eq. (12).

$$\lambda = \begin{cases} \lambda & (E[x] < 0) \\ -\lambda & (E[x] > 0) \end{cases} \tag{12}$$

3. Flowchart

The flowchart of proposed identification is shown in Fig. 2. Estimation of instantaneous AR coefficients is conducted by state estimation of time varying AR model based on the response time series data of free damping oscillation. Furthermore, the instantaneous eigen angular frequencies are obtained from the relationship between the pole and the eigen angular frequency in AR model. The instantaneous amplitude is calculated by Hilbert transform. The nonlinear coefficients are estimated by the derived regression formula in the above section using the instantaneous eigen angular frequency and the instantaneous amplitude. To decide on the sign of nonlinear coefficient, the average value of time series is calculated. Moreover, the sign of nonlinear coefficient is discriminated by Eq. (12).

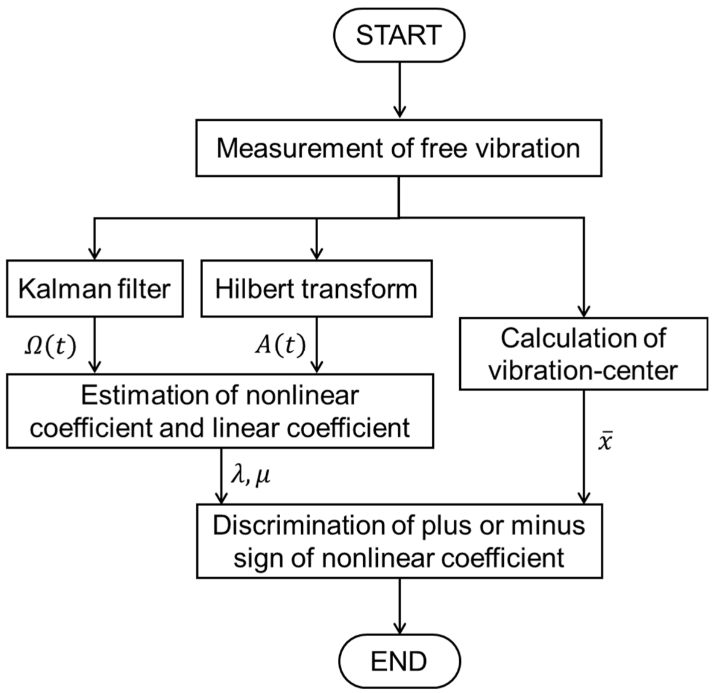


Fig. 2. Flowchart of proposed identification algorithm.

4. Numerical identification

In this section, the result of numerical identification experiment is described. First, the numerical solution of nonlinear equation of motion is calculated by the fourth Runge-Kutta method. Furthermore, the unknown parameters are estimated based on the proposed identification algorithm using the above numerical solutions.

4.1 Conditions

The calculation conditions are shown in Table 1. Here, $dt = 0.05$ is the sampling time, $X_0 = 15$ and $V_0 = 0$ are the initial conditions, $m = 1$ is the mass, $k = 1$ is the spring constant (i.e. the linear eigen angular frequency is $\omega_0 = 1$), $\zeta = 0.01$ is the damping ratio, and $\lambda = 0.01$ and $\mu = 0.01$ are second and third-order nonlinear spring coefficients, respectively.

The time histories of free oscillation are shown in Fig. 3 and the characteristics between restoring force and displacement are shown in Fig.4. The characteristic of asymmetric restoring force is observed in the region between tensile and compression sides.

Table 1 Numerical condition of nonlinear parameters and initial conditions

dt	ω_0	ζ	λ	μ	X_0	V_0
0.05	1.0	0.01	0.01	0.01	15	0

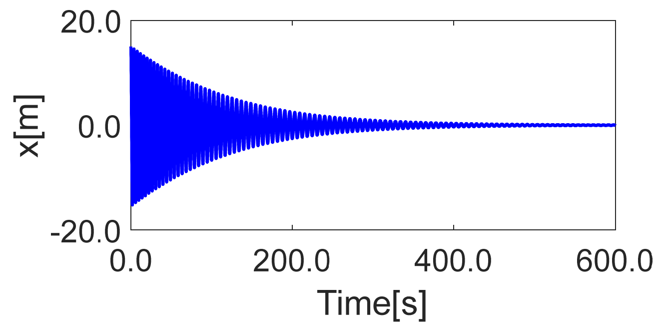


Fig. 3. Time histories of free vibration in case of positive second-order nonlinear spring coefficient and positive third-order nonlinear spring coefficient.

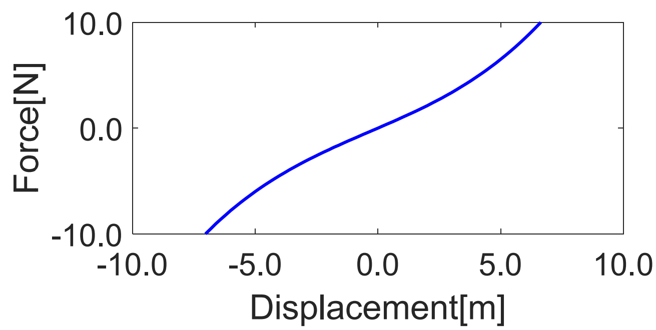


Fig.4 Characteristic of restoring force-deformation relationship in case of positive second-order nonlinear spring coefficient and positive third-order nonlinear spring coefficient.

4.2 Results and discussion

The estimation result of instantaneous eigen frequency using Kalman filter is shown in Fig 5. Here, the vertical and horizontal axes represent the eigen frequencies and, time, respectively. The calculation conditions are as follows: the time series model uses time-varying AR(M) model (here, M = 14), the covariance of system noise is $\sigma_v^2 = 1$, the covariance of observation noise is $\sigma_w^2 = 10^{-4}$, the initial value of error covariance matrix is $P[0] = 10^3\boldsymbol{I}$ (here, $\boldsymbol{I}$ represents the unit matrix of $M \times M$ dimension) and the initial value of unknown AR coefficient vector is $\boldsymbol{\theta}[0] = \boldsymbol{0}$ (here, $\boldsymbol{0}$ represents the zero vector of $1 \times M$ dimension). Here, the black dotted line represents the region of time series that used identification algorithm.

The estimated value of instantaneous eigen frequency gradually decreases. Moreover, it converges to a constant value of eigen frequency. The eigen frequency of linear system is 0.159 Hz, and, the converged eigen frequency is estimated as the 0.161 Hz. Therefore, the estimated converged eigen frequency is in good agreement with the linear eigen frequency.

The estimation result of relationship between restoring force and displacement is shown in Fig 6. Here, the vertical and horizontal axes represent the restoring force and displacement, respectively. In addition, the black and blue solids show the cases of true and estimated values, respectively. Furthermore, the estimated coefficient is shown in Table 2.

Here, the double adjusted coefficient of determination in the relationship between restoring force and displacement is 0.986. Hence, the operation of proposed identification algorithm is confirmed. This is because the identification result in the relationship between restoring force and displacement is obtained using estimated parameter.

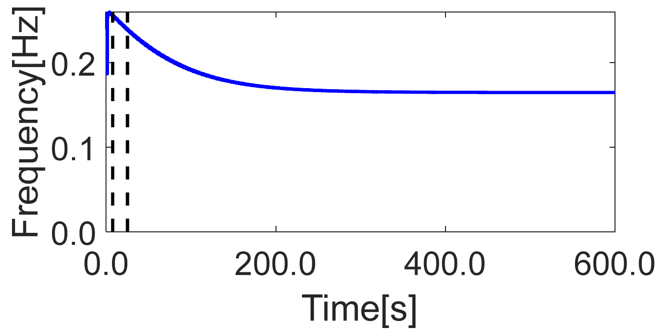


Fig. 5. Estimation result of instantaneous frequency in case of including the second and third order nonlinear spring.

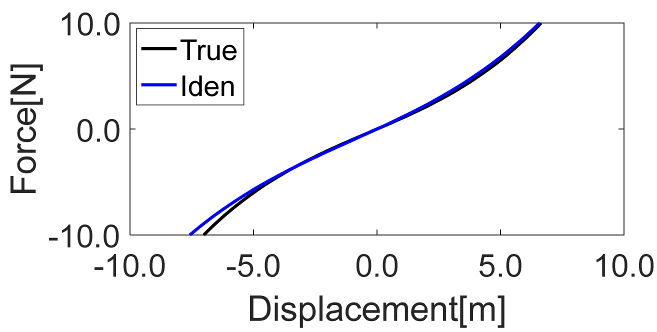


Fig. 6. Estimation result of relationship between restoring force and displacement.

Table 2 Estimation results of linear parameter and nonlinear parameters

	ω_0	λ	μ
True	1.0000	0.0100	0.0100
Estimation	1.0304	0.0182	0.0068

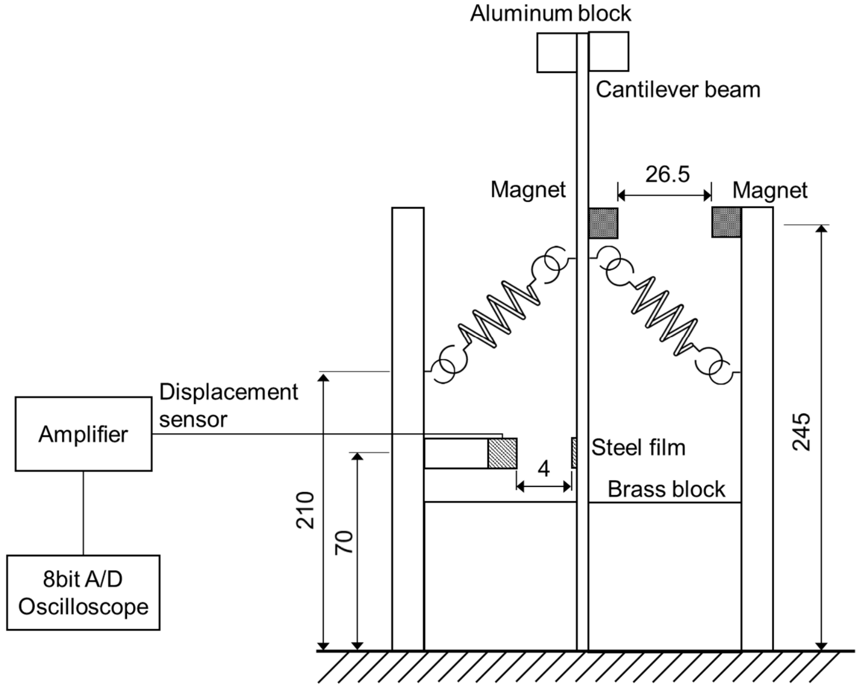
5. Experimental identification

5.1 Experimental setup

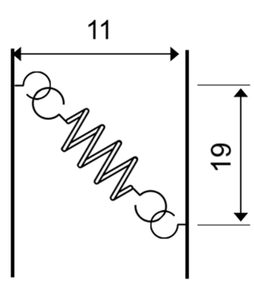
The experimental setup of the cantilever system is shown in Fig.7(a). Here, the dimension of beam is as follows: the plate is made of Aluminum material; width is 10 mm, thickness is 2 mm and length is 238 mm. The cantilever has two concentrated mass of Aluminum material at the free end, while the fixed end of cantilever is tightened by brass material block (50 mm × 50 mm × 50 mm). The second-order nonlinear restoring force is realized by magnetic force of opposed Neodymium magnets at the

245 mm from the ground along the beam length direction. The gap between opposed Neodymium magnets is 26.5 mm, and the magnetic flux density of Neodymium magnet is 240 mT. Furthermore, the third-order nonlinear restoring force is realized by the two symmetric springs located at an angle of 30º and 210 mm from the ground along the beam length direction. The detailed arrangement of the coil spring is shown in Fig.7(b). The specification of coil spring is as follows: length is 15 mm, wire diameter is 0.25 mm, coil diameter is 25 mm, number of turns is 35 and spring constant is 0.07133 N/mm.

The free damping vibration is measured using the displacement sensor under the initial displacement of 15 mm. The eddy current displacement sensor used is EX-614V of Keyence Corporation. The digital oscilloscope used is DSO QUAD of Seeed Studio Ltd. Here, the sampling frequency is 300 Hz, the bit number of A/D converter is 8 bits. The displacement sensor is located 70 mm from the ground along the beam length direction. The air gap between the displacement sensor and the cantilever is 4 mm. Here, the thin plate of quenched ribbon steel (the thickness is 3 mm) is introduced to the Aluminum cantilever to increase the sensitivity of displacement sensor.



(a) Overview of cantilever system subjected to the magnetic force, geometrical nonlinear restoring force



(b) Detail of spring geometry

Fig. 7. Experimental setup for verify the proposed nonlinear system identification.

The relationship between the restoring force and the displacement is measured by the spring scale. Therefore, the calibration from the deflection at the measurement position to the deflection at the top of cantilever is required. The calibration formula is derived based on the method of influence coefficient. Here, l, l_1, y_1 and y_2 represent the cantilever length, cantilever length of measurement position from the ground along the beam length direction, displacement at the measurement position, and displacement at the top of cantilever, respectively. In addition, Young modulus is E and the second moment of area is I. Here, the influence coefficient matrix is as follows Eq. (13):

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} l_1^3/3 & l_1^2\{2l_1 + 3(l - l_1)\}/6 \\ l_1^2\{2l_1 + 3(l - l_1)\}/6 & l^3/3 \end{bmatrix} \begin{bmatrix} 0 \\ P \end{bmatrix} \tag{13}$$

Furthermore, the calibration formula is obtained by Eq. (14).

$$y_2 = \frac{2l^3}{l_1^2\{2l_1 + 3(l - l_1)\}} y_1 \tag{14}$$

5.2 Results and discussion

The time histories of free oscillation are shown in Fig 8. Here, the vertical and horizontal axes represent the displacement at the top of cantilever and, the time, respectively. The free damping oscillation comprises the component of changing vibration center. Thus, the cantilever system confirms the subjecting to the nonlinear restoring force.

The estimation result of instantaneous eigen frequency is shown in Fig 9. Here, the vertical and horizontal axes represent the eigen frequency and, the time, respectively. The calculation conditions are as follows: the time series model uses time-varying AR(M) model (here, M = 14), the covariance of system noise is $\sigma_v^2 = 1$, the covariance of observation noise is $\sigma_w^2 = 10^{-4}$, the initial value of error covariance matrix is $P[0] = 10^3\boldsymbol{I}$ (here, $\boldsymbol{I}$ shows the unit matrix of $M \times M$ dimension), and the initial value of unknown AR coefficient vector is $\boldsymbol{\theta}[0] = \boldsymbol{0}$ (here, $\boldsymbol{0}$ denotes the zero vector of $1 \times M$ dimension). The estimated value of instantaneous eigen frequency gradually increases respect to the time evolution. Furthermore, instantaneous eigen frequency converges to a constant value.

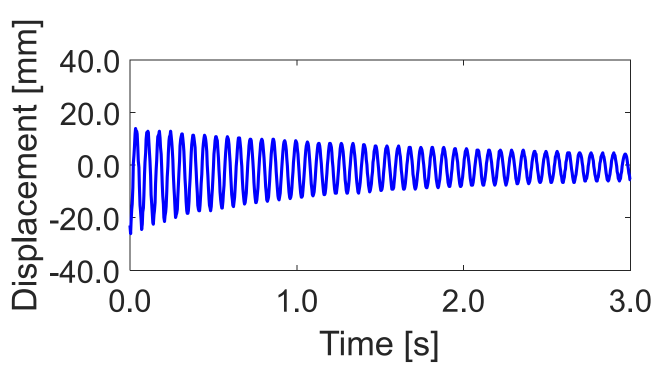


Fig. 8. Time histories of nonlinear free vibration. The slightly changes of instantaneous frequency are observed. Furthermore, the vibration center is changed from negative region to zero center.

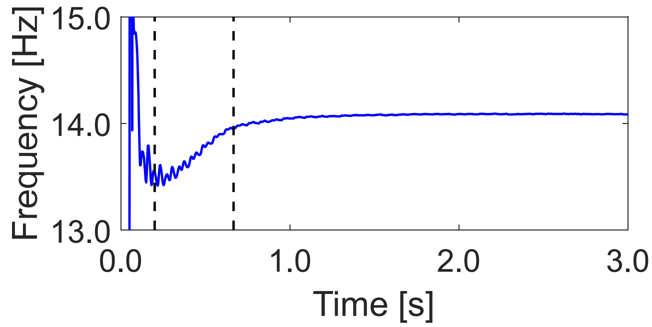


Fig. 9. Estimation result of instantaneous frequency using parameter estimation of time-varying AR model.

The estimated values of unknown linear and nonlinear parameter are shown in Table 3. Furthermore, the comparison between the estimated and experimental values using the spring scale in the relationship between restoring force and displacement is shown in Fig 10. Here, the vertical and horizontal axes represent the restoring force and the displacement, respectively. The black circle denotes the case of true values, while the blue solid curve denotes the case of estimated values. The estimation curve is in good agreement with the experimental values. Here, the double adjusted coefficient of determination in the relationship between restoring force and displacement is 0.997.

Table 3 Estimation results of linear parameter and nonlinear parameters

	ω_0	λ	μ
Estimation	97.3138	410.5332	25.6413

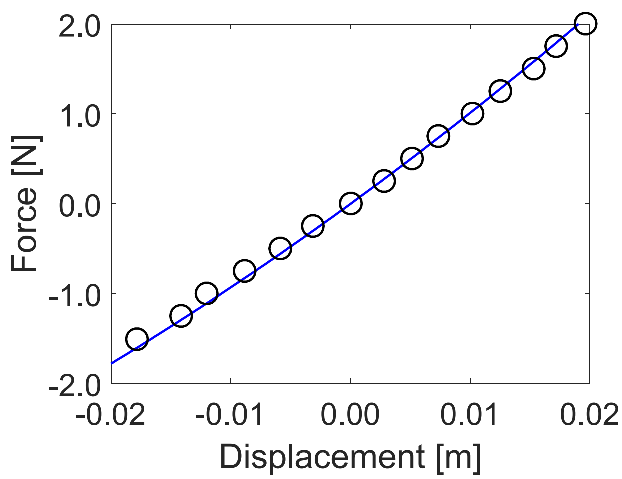


Fig. 10. Estimation result of the relationship between the restoring force and deformation. The black circle and blue solid curve represent the measurement values and, the estimation result using the proposed method, respectively.

6. Conclusion

In this paper, the nonlinear stiffness identification method based on AR time series analysis was discussed, and the following results were obtained.

(1) Identification algorithm for obtaining nonlinear spring coefficients was derived by analytical solution based on KBM method.

(2) Numerical identification experiment was conducted using the fourth Runge-Kutta method, and we found that the estimated values of nonlinear spring constant were in good agreement with the true values.

(3) Experimental identification was conducted using the cantilever system, which was simultaneous subjected to the asymmetric magnetic force and the geometrical nonlinear spring force. We obtained that the estimated values of the relationship between restoring force and displacement were in good agreement with the true experimental values, which was measured by the spring scale.

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